Concurrent monads as monads in a 2-category

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Concurrency and effects

- In algebraic semantics of programming, *Kleene algebras* axiomatize sequential composition together with nondeterministic choice and finite iteration (for linear time).

- *Concurrent Kleene algebras* (Hoare et al. 2009/11) add parallel composition.

- In categorical semantics of effectful programming, *monads* (via their Kleisli categories) axiomatize sequential composition of effectful functions.
- but they do not axiomatize parallel composition.

- Could we add parallel composition, taking inspiration from concurrent Kleene algebras?
- Spoiler: Yes, with *concurrent monads* (Rivas, Jaskelioff 2019).

This talk

- Concurrent monoids
- Concurrent monads
- Concurrent monads = monads in a certain 2-category
- Where does Kleisli fit?

Concurrent monoids

(Gischer 1984/88; Hoare et al. 2009/11)

- A *concurrent monoid* is an ordered set $(M, \leq)$ carrying two ordered monoid structures $(\mathrm{id}, (;))$ and $(\mathrm{jd}, \|)$ related by interchange inequations

$$\begin{aligned} \mathrm{id} &\leq \mathrm{jd} \\ \mathrm{jd} \mathbin{;} \mathrm{jd} &\leq \mathrm{jd} \\ \mathrm{id} &\leq \mathrm{id} \parallel \mathrm{id} \\ (k \parallel \ell) \mathbin{;} (m \parallel n) &\leq (k \mathbin{;} m) \parallel (\ell \mathbin{;} n) \end{aligned}$$

- If the 1st inequation holds as an equality, the concurrent monoid is said to be *normal*.

- If $\leq$ is $=$, we have a *duoid*, which the same as a commutative monoid.

- Classically, one requires commutativity of $\|$ and normality. We don't.

Example: Idempotent semirings

- Any idempotent semiring $(S, 0, +, 1, \times)$ comes with an order $\leq$ defined by $m \leq n$ iff $m + n = n$.

- It induces two concurrent monoids, $(S, \leq, 0, +, 1, \times)$ and $(S, \geq, 1, \times, 0, +)$.

  In the first, $(;)$ is commutative and idempotent.

  In the second, $\|$ is commutative and idempotent.

- E.g., $(\mathbb{N}, \leq, 0, \max, 0, +)$ (for space) and $(\mathbb{N}, \geq, 0, +, 0, \max)$ (for time).

  These concurrent monoids are normal.

- E.g., the Boolean semiring $(\mathbb{B}, \supset, \mathrm{ff}, \vee, \mathrm{tt}, \wedge)$.

  This concurrent monoid is not normal.

Example: Finite (un)weighted word languages

- Given a set Σ (an alphabet).

- The ordered set $(\mathcal{M}_{\mathrm{f}}(\mathsf{List}\Sigma), \subseteq)$ (finited weighted word languages) carries a concurrent monoid structure $(1, \cdot, 1, ⧢)$ where
  - $1 = \{[]\}$,
  - $L \cdot L' = \{u +\!\!+ v \mid u \in L, v \in L'\}$ (concatenation),
  - $L ⧢ L' = \bigcup\{u ⧢ v \mid u \in L, v \in L'\}$ (shuffle).

- $\mathcal{M}_{\mathrm{f}}$ can be replaced with $\mathcal{P}_{\mathrm{f}}$ (for unweighted languages) or the free S-semimodule monad (for S-weighted languages).

- Words can be replaced with Mazurkiewicz traces, labelled posets.

Concurrent monads

(Rivas, Jaskelioff 2019)

- A *concurrent monad* on an ordered monoidal category $(\mathbb{C}, \leq, \mathsf{I}, \otimes)$ is an ordered endofunctor T carrying both
  - an ordered monad structure (η, μ) and
  - an ordered lax monoidal functor structure (ψ^0, ψ)

  satisfying interchange inequations

$$\begin{tikzcd}
\mathsf{I} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\geq"] & \mathsf{I} \arrow[d, "\eta_I"] \\
\mathsf{I} \arrow[r, "\psi^0"] & T\mathsf{I}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{I} \arrow[r, "\psi^0"] \arrow[d, equal] \arrow[drr, phantom, "\geq"] & T\mathsf{I} \arrow[r, "T\psi^0"] & T(T\mathsf{I}) \arrow[d, "\mu_I"] \\
\mathsf{I} \arrow[rr, "\psi^0"] & & T\mathsf{I}
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes Y \arrow[r, equal] \arrow[d, "\eta_X \otimes \eta_Y"'] \arrow[dr, phantom, "\geq"] & X \otimes Y \arrow[d, "\eta_{X \otimes Y}"] \\
TX \otimes TY \arrow[r, "\psi_{X,Y}"] & T(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
T(TX) \otimes T(TY) \arrow[r, "\psi_{TX,TY}"] \arrow[d, "\mu_X \otimes \mu_Y"'] \arrow[drr, phantom, "\geq"] & T(TX \otimes TY) \arrow[r, "T\psi_{X,Y}"] & T(T(X \otimes Y)) \arrow[d, "\mu_{X \otimes Y}"] \\
TX \otimes TY \arrow[rr, "\psi_{X,Y}"] & & T(X \otimes Y)
\end{tikzcd}$$

- If $\leq$ is $=$, this reduces to a *lax monoidal monad*, which is the same as a *commutative monad*.

Kleisli for a concurrent monad

- Given a concurrent monad $(T, \eta, \mu, \psi^0, \psi)$ on an ordered monoidal category $(\mathbb{C}, \leq, \mathsf{I}, \otimes)$.
- Recall that the Kleisli ordered category and adjoints
$$\begin{tikzcd} \mathbb{K} \arrow[d, bend left, "K"] \arrow[d, phantom, "\dashv"] \\ \mathbb{C} \arrow[u, bend left, "J"] \end{tikzcd}$$
of (T, η, μ) are given by these data:
    - $|\mathbb{K}| = |\mathbb{C}|$, $\mathbb{K}(X, Y) = \mathbb{C}(X, TY)$,
    - $k \leq^{\mathsf{K}} \ell$ in $\mathbb{K}(X, Y)$ iff $k \leq \ell$ in $\mathbb{C}(X, TY)$,
    - $\mathrm{id}^{\mathsf{K}}{}_X = \; X \xrightarrow{\eta_X} TX$,
    $k \,;^{\mathsf{K}} \ell = \; X \xrightarrow{k} TY \xrightarrow{T\ell} T(TZ) \xrightarrow{\mu_Z} TZ$
    for $k \in \mathbb{K}(X, Y)$, $\ell \in \mathbb{K}(Y, Z)$,
    - $JX = X$, $Jf = \; X \xrightarrow{f} Y \xrightarrow{\eta_Y} TY$ for $f \in \mathbb{C}(X, Y)$,
    - $KX = TX$, $Kk = \; TX \xrightarrow{Tk} T(TY) \xrightarrow{\mu_Y} TY$ for $k \in \mathbb{K}(X, Y)$,
    - the unit is η_X, the counit is $\varepsilon_X = \mathrm{id}_{TX}$.

Kleisli for a concurrent monad ctd

- In addition, ψ^0 and ψ induce these data:
  - $\mathrm{jd} = \ \mathsf{I} \xrightarrow{\psi^0} T\mathsf{I}$

    $k \parallel \ell = \ X \otimes U \xrightarrow{k \otimes \ell} TY \otimes TV \xrightarrow{\psi_{Y,V}} T(Y \otimes V)$

    for $k \in \mathbb{K}(X, Y)$, $\ell \in \mathbb{K}(U, V)$.
- They satisfy these (in)equations:

$$\begin{gathered}
(\mathrm{jd} \parallel k) \,;^{\mathsf{K}}\, J\lambda_Y = J\lambda_X \,;^{\mathsf{K}}\, k \\
k \,;^{\mathsf{K}}\, J\rho_Y = J\rho_X \,;^{\mathsf{K}}\, (k \parallel \mathrm{id}) \\
((k \parallel \ell) \parallel m) \,;^{\mathsf{K}}\, J\alpha_{Y,U,W} = J\alpha_{X,Z,V} \,;^{\mathsf{K}}\, (k \parallel (\ell \parallel m))
\end{gathered}$$

$$\begin{gathered}
{\mathrm{id}^{\mathsf{K}}}_{\mathsf{I}} \leq^{\mathsf{K}} \mathrm{jd} \\
\mathrm{jd} \,;^{\mathsf{K}}\, \mathrm{jd} \leq^{\mathsf{K}} \mathrm{jd} \\
{\mathrm{id}^{\mathsf{K}}}_{X \otimes Y} \leq^{\mathsf{K}} {\mathrm{id}^{\mathsf{K}}}_X \parallel {\mathrm{id}^{\mathsf{K}}}_Y \\
(k \parallel \ell) \,;^{\mathsf{K}}\, (m \parallel n) \leq^{\mathsf{K}} (k \,;^{\mathsf{K}}\, m) \parallel (\ell \,;^{\mathsf{K}}\, n)
\end{gathered}$$

$$\begin{gathered}
J(f \otimes g) \,;^{\mathsf{K}}\, (m \parallel n) = (Jf \,;^{\mathsf{K}}\, m) \parallel (Jg \,;^{\mathsf{K}}\, n) \\
(k \parallel \ell) \,;^{\mathsf{K}}\, J(h \otimes i) = (k \,;^{\mathsf{K}}\, Jh) \parallel (\ell \,;^{\mathsf{K}}\, Ji)
\end{gathered}$$

- This like a concurrent monoid but “typed”!

Example: State

- We consider examples on the ordered Cartesian monoidal closed category $(\mathsf{Poset}, \leq, 1, \times, \Rightarrow)$
  which has as objects ordered sets, as maps monotone functions,
  $f \leq g$ for $f, g : X \to Y$ iff $fx \leq^Y gx$ for all x.

- The writer/reader/state monads from FP are concurrent monads for parameters with suitable structure.

- $TX = S \Rightarrow S \times X \qquad$ for $(S, \top, \wedge)$ a lower semilattice

- $\eta_X\, x = \lambda s.\, (s, x)$
  $\mu_X\, f = \lambda s.\, \mathrm{let}\ (s', g) = f\, s\ \mathrm{in}\ g\, s'$
- $\psi^0\, \star = \lambda\_.\, (\top, \star)$
  $\psi_{X,Y}\, (f, g) = \lambda s.\, \mathrm{let}\ ((s_0, x), (s_1, y)) = (f\, s, g\, s)\ \mathrm{in}\ (s_0 \wedge s_1, (x, y))$

- This concurrent monad is not normal ($\eta_1 \neq \psi^0$).

- A computation is one big atomic step.
- In parallel composition, the two given atomic steps happen "simultaneously", the competing writes are reconciled by $\wedge$.

Example: Resumptions for state shared in interleaving

- $TX = \mu Z.\ \underbrace{X}_{ret}\ \underbrace{+}_{or}\ (\underbrace{S\ \Rightarrow}_{grab}\ \underbrace{\mathsf{List}}_{branch}\ (\underbrace{S\ \times}_{yield}\ \underbrace{Z}_{repeat}))$
  for S discretely ordered (for simplicity)
- $\leq^{TX}$ is induced by $\leq^{\mathsf{List}\,Y}$, in turn induced by order-preserving inclusion between lists

- $\eta\, x = ret\, x$
  $\mu\,(ret\, r) = r$
  $\mu\,(grab\, k) = grab\,(\lambda s.\,[(s', \mu_X\, r \mid (s', r) \leftarrow k\, s])$
- $\psi^0 \star = ret\,\star$
  $\psi\,(ret\, x, ret\, y) = ret\,(x, y)$
  $\psi\,(ret\, x, grab\,\ell) = grab\,(\lambda s.\,[(s', \psi\,(ret\, x, r)) \mid (s', r) \leftarrow \ell\, s])$
  $\psi\,(grab\, k, ret\, y) = grab\,(\lambda s.\,[(s', \psi\,(r, ret\, y)) \mid (s', r) \leftarrow k\, s])$
  $$\psi\,(grab\, k, grab\,\ell) = grab\,(\lambda s.\,[(s', \psi\,(r, grab\,\ell)) \mid (s', r) \leftarrow k\, s] \\ +\!\!+[(s', \psi\,(grab\, k, r)) \mid (s', r) \leftarrow \ell\, s])$$

- Although we use List rather than $\mathcal{M}_{\mathrm{f}}$ or $\mathcal{P}_{\mathrm{f}}$, associativity of ψ and the 4th interchange inequation hold.

- A computation consists of small atomic steps. In parallel composition, these are interleaved.

Example: Bags of traces for state shared in interleaving

- $TX = \underbrace{\mathcal{M}_{\mathrm{f}}}_{\mathit{BRANCH}} (T'X)$ where $T'X = \mu Z.\, \underbrace{X}_{\mathit{ret}} \underbrace{+}_{\mathit{or}} \underbrace{S \times}_{\mathit{grab}} \underbrace{S \times}_{\mathit{yield}} \underbrace{Z}_{\mathit{repeat}}$
- $\leq^{TX}$ is induced by $\leq^{\mathcal{M}_{\mathrm{f}} Y}$, induced by multiset inclusion
- $\eta\, x = \{\mathit{ret}\ x\}$

  $\mu\ \mathit{ts} = \bigcup (\mathcal{M}_{\mathrm{f}}\ \mu'\ \mathit{ts})$ where

  $$\begin{aligned}
  &\mu' : T'(TX) \to TX\\
  &\mu'\,(\mathit{ret}\ \mathit{ts}) = \mathit{ts}\\
  &\mu'\,(\mathit{grab}\ s\ s'\ t) = T'\,(\mathit{grab}\ s\ s')\,(\mu'\ t)
  \end{aligned}$$
- $\psi^0 \star = \{\mathit{ret}\ \star\}$

  $\psi\,(\mathit{ts}_0, \mathit{ts}_1) = \bigcup\{t_0 \sqcup\!\sqcup\ t_1 \mid t_0 \leftarrow \mathit{ts}_0, t_1 \leftarrow \mathit{ts}_1\}$ where

  $$\begin{aligned}
  &\sqcup\!\sqcup : T'X \times T'Y \to T(X \times Y)\\
  &\mathit{ret}\ x \sqcup\!\sqcup\ \mathit{ret}\ y = \{\mathit{ret}\,(x, y)\}\\
  &\mathit{ret}\ x \sqcup\!\sqcup\ \mathit{grab}\ s\ s'\ t = T'\,(\mathit{grab}\ s\ s')\,(\mathit{ret}\ x \sqcup\!\sqcup\ t)\\
  &\mathit{grab}\ s\ s'\ t \sqcup\!\sqcup\ \mathit{ret}\ y = T'\,(\mathit{grab}\ s\ s')\,(t \sqcup\!\sqcup\ \mathit{ret}\ y)\\
  &\mathit{grab}\ s_0\ s_0'\ t_0 \sqcup\!\sqcup\ \mathit{grab}\ s_1\ s_1'\ t_1 = T'\,(\mathit{grab}\ s_0\ s_0')\,(t_0 \sqcup\!\sqcup\ \mathit{grab}\ s_1\ s_1'\ t_1)\\
  &\qquad\qquad\qquad\qquad \cup T'\,(\mathit{grab}\ s_1\ s_1')\,(\mathit{grab}\ s_0\ s_0'\ t_0 \sqcup\!\sqcup\ t_1)
  \end{aligned}$$

Concurrent monads as concurrent monoids

- Monads on a category $\mathbb{C}$ are the same as monoids in the monoidal category $([\mathbb{C}, \mathbb{C}], \mathsf{Id}, \cdot)$ of endofunctors on $\mathbb{C}$ and composition.

- Classical monoids are monoids in the monoidal category $(\mathsf{Set}, 1, \times)$.

- (Accessible) concurrent monads on an ordered monoidal category $(\mathbb{C}, \leq, \mathsf{I}, \otimes)$ are the same as concurrent monoids in the (lp) ordered duoidal category $([\mathbb{C}, \mathbb{C}]_{\mathsf{a}}, \leq, \mathsf{Id}, \cdot, \mathsf{Jd}, \star)$ of (accessible) ordered endofunctors on $\mathbb{C}$, composition and Day convolution.

- (Accessible) lax monoidal monads on a monoidal category $(\mathbb{C}, \mathsf{I}, \otimes)$ are duoids in the (lp) duoidal category $([\mathbb{C}, \mathbb{C}]_{\mathsf{a}}, \mathsf{Id}, \cdot, \mathsf{Jd}, \star)$ of (accessible) endofunctors on $\mathbb{C}$.

- Classical concurrent monoids are concurrent monoids in the ordered duoidal category $(\mathsf{Poset}, \leq, 1, \times, 1, \times)$.

Concurrent monads as monads in a 2-category?

- Monads (together with their underlying categories) are the same as monads in the 2-category Cat.

- Lax-monoidal monads (= commutative monads) are the same as monads in the 2-category $\mathbf{MonCat}_\ell$ of monoidal categories, lax-monoidal functors and monoidal natural transformations.

- Can concurrent monads can be characterized as monads in a suitable 2-category?

Concurrent monads as monads in a 2-category

- Consider the 2-category with
  - as objects, ordered monoidal categories,
  - as 1-morphisms, ordered lax-monoidal functors,
  - as 2-morphisms, oplax-monoidal natural transformations
- Here we call a nat transf τ between ordered lax-monoidal functors (F, ϕ^0, ϕ), $(G, \phi^{0\prime}, \phi') : (\mathbb{C}, \leq, \mathsf{I}, \otimes) \to (\mathbb{D}, \leq', \mathsf{I}', \otimes')$ *oplax-monoidal* if

$$\begin{tikzcd}
\mathsf{I}' \arrow[r, "\phi^0"] \arrow[d, equal] \arrow[dr, phantom, "\geq'"] & F\mathsf{I} \arrow[d, "\tau_{\mathsf{I}}"] \\
\mathsf{I}' \arrow[r, "\phi^{0\prime}"] & G\mathsf{I}
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \otimes' FY \arrow[r, "\phi_{X,Y}"] \arrow[d, "\tau_X \otimes' \tau_Y"'] \arrow[dr, phantom, "\geq'"] & F(X \otimes Y) \arrow[d, "\tau_{X \otimes Y}"] \\
GX \otimes' GY \arrow[r, "\phi'_{X,Y}"] & G(X \otimes Y)
\end{tikzcd}$$

- Concurrent monads are precisely monads in this 2-category:

  The interchange inequations state exactly that η, μ are oplax-monoidal.

A closer look at Kleisli

- $(\mathbb{K}, \leq)$ is an lax-functorially ordered monoidal category in that $\mathsf{I}^{\mathsf{K}} = (\mathsf{I}, \mathsf{jd})$, $\otimes^{\mathsf{K}} = (\otimes, \|)$ are lax-functors satisfying unitality and associativity.

- (J, ι^0, ι) where $\iota^0 = \mathrm{id}^{\mathsf{K}}{}_I$ and $\iota_{X,Y} = \mathrm{id}^{\mathsf{K}}{}_{X \otimes Y}$ is an oplax-naturally ordered strict-monoidal functor
  in that ι^0 and ι are oplax-natural transformations.

- (K, κ^0, κ) where $\kappa^0 = \psi^0$ and $\kappa_{X,Y} = \psi_{X,Y}$ is an oplax-naturally ordered lax-monoidal functor
  in that κ^0 and κ are oplax-natural transformations.

- The unit η of the adjunction is oplax-monoidal natural.

- The counit ε of the adjunction, $\varepsilon_X = \mathrm{id}_{TX}$, is monoidal natural.

Lax-functorially ordered monoidal categories?

- To state the coherence conditions of a lax-functorially ordered monoidal category, one needs to go unbiased.

- A lax-functorially ordered monoidal category $(\mathbb{C}, \leq)$ has
  - lax-functors typed $(X_1 \otimes \ldots \otimes X_n)$,
  - natural transformations $\iota_X : X \to (X)$,
  - natural transformations $\gamma^{k_1,\ldots,k_n}_{X_{11},\ldots,X_{nk_n}}$: $((X_{11} \otimes \ldots \otimes X_{1k_1}) \otimes \ldots \otimes (X_{n1} \otimes \ldots \otimes X_{nk_n})) \to (X_{11} \otimes \ldots \otimes X_{nk_n})$

  satisfying coherence inequations à la

$$\begin{tikzcd}
(((X) \otimes (Y \otimes Z)) \otimes ((U))) \arrow[r, "\gamma^{2,1}_{(Z),(Y\otimes Z),(U)}"] \arrow[d, "(\gamma^{1,2}_{X,Y,Z} \otimes \gamma^{1}_{U})"'] \arrow[dr, phantom, "\geq"] & ((X) \otimes (Y \otimes Z) \otimes (U)) \arrow[d, "\gamma^{1,2,1}_{X,Y,Z,U}"] \\
((X \otimes Y \otimes Z) \otimes (U)) \arrow[r, "\gamma^{3,1}_{X,Y,Z,U}"'] & (X \otimes Y \otimes Z \otimes U)
\end{tikzcd}$$

Oplax-naturally ordered lax-monoidal functors?

- A similar problem arises with oplax-naturally ordered lax-monoidal functors.

- An oplax-naturally ordered lax-monoidal functor F has
  - oplax-natural transformations $\phi^n_{X_1,\ldots,X_n} : (FX_1 \otimes \ldots \otimes FX_n) \to F(X_1 \otimes \ldots \otimes X_n)$

  satisfying coherence conditions like

$$\begin{tikzcd}
((FX \otimes FY) \otimes (FZ)) \arrow[rr, "\gamma^{2,1}_{FX,FY,FZ}"] \arrow[d, "(\phi^2_{X,Y} \otimes \phi^1_Z)"'] & \arrow[d, phantom, "\geq"] & (FX \otimes FY \otimes FZ) \arrow[d, "\phi^3_{X,Y,Z}"] \\
(F(X \otimes Y) \otimes F(Z)) \arrow[r, "\phi^2_{(X \otimes Y),(Z)}"'] & F((X \otimes Y) \otimes (Z)) \arrow[r, "F\gamma^{2,1}_{X,Y,Z}"'] & F(X \otimes Y \otimes Z)
\end{tikzcd}$$

A bigger 2-category

- The Kleisli ordered category and adjoints of (the underlying ordered monad of) a concurrent monad do not fit into the above 2-category.

- They fit into a bigger 2-category with
  - as objects, ordered lax-functorially monoidal categories,
  - as 1-morphisms, ordered oplax-naturally lax-monoidal functors,
  - as 2-morphisms, oplax-monoidal natural transformations

- (We could relax the concept of concurrent monad to a monad in this 2-category, but do not know proper examples. . .)

- Differently, the Kleisli category and adjoints of the underlying monad of a lax-monoidal monad fit into the same 2-category $\mathbf{MonCat}_\ell$ where it is a monad. They make its Kleisli object.

Takeaway

- Parallel composition is axiomatizable relatively smoothly for typed effectful computation.
- Ordered category theory streamlines the development, but there are many subtleties.
- Resumption models (for branching time) can be captured well, with room for variations.
- Plenty to do on the front of examples and applications, eg true concurrency models.

- The theory gets involved, but very interesting.

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EXTRA SLIDES

Concurrent monoids ctd

- The 4th inequation implies the 1st.
  So the 1st is redundant.

- The 1st inequation implies the converses of the 2nd and 3rd inequations.
  So the 2nd and 3rd actually hold as equalities for any concurrent monoid.

- If a concurrent monoid is normal (the 1st inequation holds as an equality), then $k \,;\, \ell \leq k \parallel \ell$ and $\ell \,;\, k \leq k \parallel \ell$.

Example: A 3-valued logic

- $(\{\mathrm{ff}, \mathrm{u}, \mathrm{tt}\}, \supset, \mathrm{ff}, \vee, \mathrm{tt}, \wedge)$ is a concurrent monoid where

  $\supset$ is generated by $\mathrm{ff} \supset m$, $m \supset \mathrm{tt}$ and

  $\vee$ and $\wedge$ are defined by

$\vee$	ff	u	tt
ff	ff	u	tt
u	u	tt	tt
tt	tt	tt	tt

$\wedge$	ff	u	tt
ff	ff	ff	ff
u	ff	ff	u
tt	ff	u	tt

  Here, both (;) and $\|$ are nonidempotent.

Example: A 4-valued logic

- $(\{\mathsf{ff}, \mathsf{u}_\mathsf{L}, \mathsf{u}_\mathsf{R}, \mathsf{tt}\}, \supset, \mathsf{ff}, \vee, \mathsf{tt}, \wedge)$ is a concurrent monoid where
  $\supset$ is generated by $\mathsf{ff} \supset m$, $m \supset \mathsf{tt}$, $\mathsf{u}_\mathsf{L} \supset \mathsf{u}_\mathsf{R}$ and
  $\vee$ and $\wedge$ are defined by

$\vee$	ff	u_L	u_R	tt
ff	ff	u_L	u_R	tt
u_L	u_L	u_L	u_R	tt
u_R	u_R	tt	tt	tt
tt	tt	tt	tt	tt

$\wedge$	ff	u_L	u_R	tt
ff	ff	ff	ff	ff
u_L	ff	ff	u_L	u_L
u_R	ff	ff	u_R	u_R
tt	ff	u_L	u_R	tt

Here, both (;) and $\|$ are both nonidempotent as well as noncommutative.

Example: Writer

- We consider examples on the ordered Cartesian monoidal closed category $(\mathrm{Poset}, \leq, 1, \times, \Rightarrow)$ which has as objects ordered sets, as maps monotone functions, $f \leq g$ for $f, g : X \to Y$ iff $fx \leq^Y gx$ for all x.

- The writer/reader/state monads from FP are concurrent monads for parameters with suitable structure.

- $TX = M \times X$ for $(M, o, +, e, \cdot)$ a concurrent monoid

- $\eta_X\, x = (o, x)$ $\mu_X\, (m, (m', x)) = (m + m', x)$
- $\psi^0 \star = (e, \star)$ $\psi_{X,Y}\, ((m, x), (m', y)) = (m \cdot m', (x, y))$

- This concurrent monad is normal ($\eta_1 = \psi^0$) if the concurrent monoid is normal ($o = e$).

Reader

- $TX = S \Rightarrow X$
  for S any ordered set (it is ok if $\leq^S$ is $=$)
- NB! $S \Rightarrow X$ is the ordered set of monotone functions from S to X (if $\leq^S$ is $=$, all functions are monotone).

- $\eta_X \, x = \lambda \_ . \, x$
  $\mu_X \, f = \lambda s. \, f \, s \, s$
- $\psi^0 \star = \lambda \_ . \, \star$
  $\psi_{X,Y} \, (f, g) = \lambda s. \, (f \, s, g \, s)$

- The interchange inequations hold as equalities, so this concurrent monad is commutative.

Concurrent monoid objects

- To define what makes a concurrent monoid object in a general ordered category, this category has to be ordered duoidal.
- An *ordered duoidal category* is an ordered category $(\mathbb{D}, \leq)$ with two ordered monoidal structures $(I, \olessthan)$, $(J, \circledast)$ and maps

$$\begin{gathered} \iota : J \to I \\ \Delta : J \to J \olessthan J \\ \nabla : I \circledast I \to I \\ \xi : (F \olessthan G) \circledast (H \olessthan K) \to (F \circledast H) \olessthan (G \circledast K) \end{gathered}$$

natural in F, G, H, K satisfying some equations.

Concurrent monoid objects ctd

- A *concurrent monoid object* in an ordered duoidal category is an object M with two monoid structures (o, a) wrt. $(I, \olessthan)$ and (e, m) wrt. $(J, \circledast)$ satisfying interchange inequations

$$\begin{tikzcd}
J \arrow[r, "\iota"] \arrow[d, equal] \arrow[dr, phantom, "\geq"] & I \arrow[d, "o"] \\
J \arrow[r, "e"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
J \arrow[r, "\nabla"] \arrow[d, equal] & J \olessthan J \arrow[r, "e \olessthan e"] \arrow[d, phantom, "\geq"] & M \olessthan M \arrow[d, "a"] \\
J \arrow[rr, "e"] & & M
\end{tikzcd}$$

$$\begin{tikzcd}
I \circledast I \arrow[r, "\Delta"] \arrow[d, "o \circledast o"'] \arrow[dr, phantom, "\geq"] & I \arrow[d, "o"] \\
M \circledast M \arrow[r, "m"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
(M \olessthan M) \circledast (M \olessthan M) \arrow[r, "\xi"] \arrow[d, "a \circledast a"'] & (M \circledast M) \olessthan (M \circledast M) \arrow[r, "m \olessthan m"] \arrow[d, phantom, "\geq"] & M \olessthan M \arrow[d, "a"] \\
M \circledast M \arrow[rr, "m"] & & M
\end{tikzcd}$$

- A (classical) concurrent monoid is a concurrent monoid object in the ordered duoidal category $(\mathrm{Poset}, 1, \times, 1, \times)$.

Concurrent monads as concurrent monoids

- If $(\mathbb{C}, \leq, \mathsf{I}, \otimes)$ is a ordered lp monoidal category, then $([\mathbb{C}, \mathbb{C}]_{\mathsf{a}}, \leq, \mathsf{Id}, \cdot, \mathsf{Jd}, \star)$ is an ordered duoidal category.
- Here $(\mathsf{Jd}, \star)$ is the Day convolution monoidal structure

$$\begin{aligned} \mathsf{Jd} Z &= \mathbb{C}(\mathsf{I}, Z) \bullet \mathsf{I} \\ (F \star G) Z &= \int^{X,Y} \mathbb{C}(X \otimes Y, Z) \bullet (FX \otimes GY) \end{aligned}$$

- An accessible concurrent monad $(T, \eta, \mu, \psi^0, \psi)$ is the same as a concurrent monoid object in this ordered duoidal category.

$$\begin{array}{c} \mathsf{I} \to T\mathsf{I} \text{ in } \mathbb{C} \\ \hline\hline \mathbb{C}(\mathsf{I}, Z) \to \mathbb{C}(\mathsf{I}, TZ) \text{ nat in } Z \text{ in } \mathsf{Poset} \\ \hline\hline \underbrace{\mathbb{C}(\mathsf{I}, Z) \bullet \mathsf{I}}_{\mathsf{Jd} Z} \to TZ \text{ nat in } Z \text{ in } \mathbb{C} \end{array}$$

$$\begin{array}{c} TX \otimes TY \to T(X \otimes Y) \text{ nat in } X, Y \text{ in } \mathbb{C} \\ \hline\hline \mathbb{C}(X \otimes Y, Z) \to \mathbb{C}(TX \otimes TY, TZ) \text{ nat in } X, Y, Z \text{ in } \mathsf{Poset} \\ \hline\hline \mathbb{C}(X \otimes Y, Z) \bullet (TX \otimes TY) \to TZ \text{ nat in } X, Y, Z \text{ in } \mathbb{C} \\ \hline\hline \underbrace{\int^{X,Y} \mathbb{C}(X \otimes Y, Z) \bullet (TX \otimes TY)}_{(T \star T) Z} \to TZ \text{ nat in } Z \text{ in } \mathbb{C} \end{array}$$

Concurrent monads vs bistrong monads

- A bistrong monad only supports parallel composition of a map with a pure map (on the left or on the right), unless it is commutative.
- In the Kleisli category of a bistrong monad (a premonoidal category) we generally have

$$\begin{tikzcd}
X \otimes U \arrow[r, "Jf \rtimes \mathrm{id}_U"] \arrow[d, "\mathrm{id}_X \ltimes \ell"'] \arrow[dr, "f \ltimes \ell" description] & Y \otimes U \arrow[d, "\mathrm{id}_Y \ltimes \ell"] \\
X \otimes V \arrow[r, "Jf \rtimes \mathrm{id}_V"'] & Y \otimes V
\end{tikzcd}
\quad
\begin{tikzcd}
X \otimes U \arrow[r, "k \rtimes \mathrm{id}_U"] \arrow[d, "\mathrm{id}_X \ltimes \ell"'] \arrow[dr, phantom, "\neq"] & Y \otimes U \arrow[d, "\mathrm{id}_Y \ltimes \ell"] \\
X \otimes V \arrow[r, "k \rtimes \mathrm{id}_V"'] & Y \otimes V
\end{tikzcd}
\quad
\begin{tikzcd}
X \otimes U \arrow[r, "k \rtimes \mathrm{id}_U"] \arrow[d, "\mathrm{id}_X \ltimes Jg"'] \arrow[dr, "k \rtimes g" description] & Y \otimes U \arrow[d, "\mathrm{id}_Y \ltimes Jg"] \\
X \otimes V \arrow[r, "k \rtimes \mathrm{id}_V"'] & Y \otimes V
\end{tikzcd}$$

  where $\ltimes : \mathbb{C} \times \mathbb{K} \to \mathbb{K}$, $\rtimes : \mathbb{K} \times \mathbb{C} \to \mathbb{K}$.

- In the Kleisli ordered category of a concurrent monad (a concurrent category) we have

$$\begin{tikzcd}
X \otimes U \arrow[r, "k \parallel {\mathrm{id}^{\mathsf{K}}}_U"] \arrow[d, "{\mathrm{id}^{\mathsf{K}}}_X \parallel \ell"'] \arrow[dr, "k \parallel \ell" description] \arrow[dr, phantom, "\leq", shift right=5ex] \arrow[dr, phantom, "\geq", shift left=5ex] & Y \otimes U \arrow[d, "{\mathrm{id}^{\mathsf{K}}}_Y \parallel \ell"] \\
X \otimes V \arrow[r, "k \parallel {\mathrm{id}^{\mathsf{K}}}_V"'] & Y \otimes V
\end{tikzcd}$$