Symmetric List Objects

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joint work with
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Lists

In scheme:

```
(a b c) ; a list of three elements
(a . (b . (c . ()))) ; the same list
() ; the empty list
(car '(a b c)) ; a
(cdr '(a b c)) ; (b c)
```

In Haskell:

```
data [a] = [] | a : [a]
```

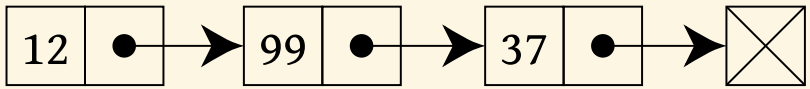
In Agda:

```
data ℒist (A : Set) : Set where
  [] : ℒist A
  _::_ : A → ℒist A → ℒist A
```

In Rocq:

```
Inductive list (A : Type) : Type :=
  | nil : list A
  | cons : A -> list A -> list A.
```

Linked lists:



Constructions

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We can also construct lists as initial algebras of the following functor:

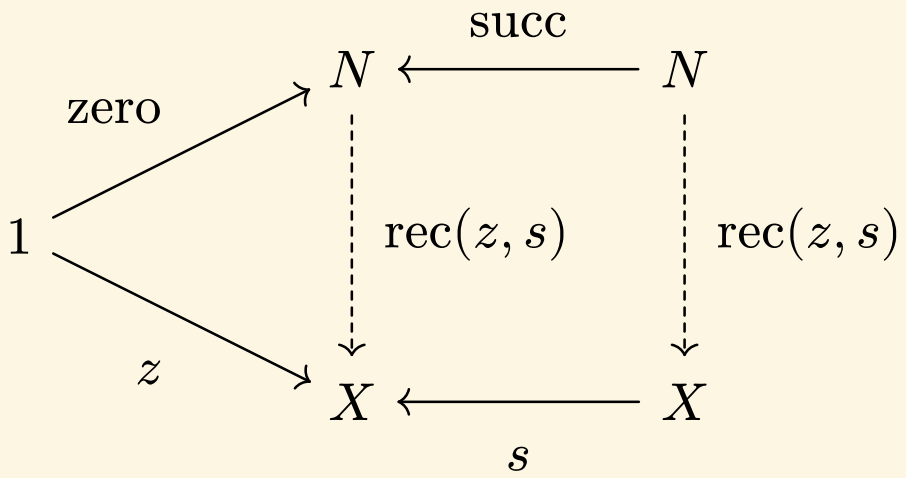
$$F(X) = 1 + A \times X$$

but this requires an initial object, and for the colimit of the following ω-chain to exist:

$$0 \to F(0) \to FF(0) \to FFF(0) \to \ldots$$

List Objects

Lawvere introduced[1] natural numbers objects in his axiomatisation of set theory (ETCS):



[1] F. W. Lawvere, "An Elementary Theory of the Category of Sets*," *Proceedings of the National Academy of Sciences*, vol. 52, no. 6, pp. 1506–1511, Dec. 1964, doi: 10.1073/pnas.52.6.1506.

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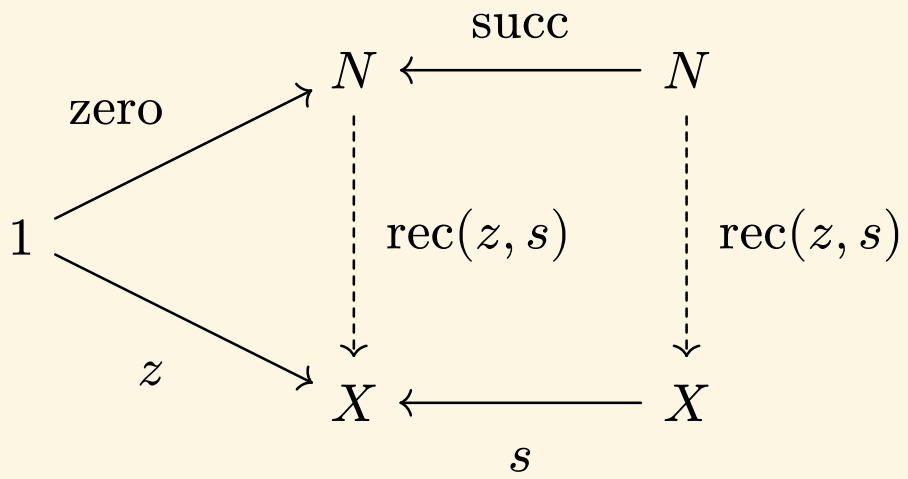

We can adapt this universal property to axiomatise list objects in general monoidal categories.

[1] F. W. Lawvere, "An Elementary Theory of the Category of Sets*," *Proceedings of the National Academy of Sciences*, vol. 52, no. 6, pp. 1506–1511, Dec. 1964, doi: 10.1073/pnas.52.6.1506.

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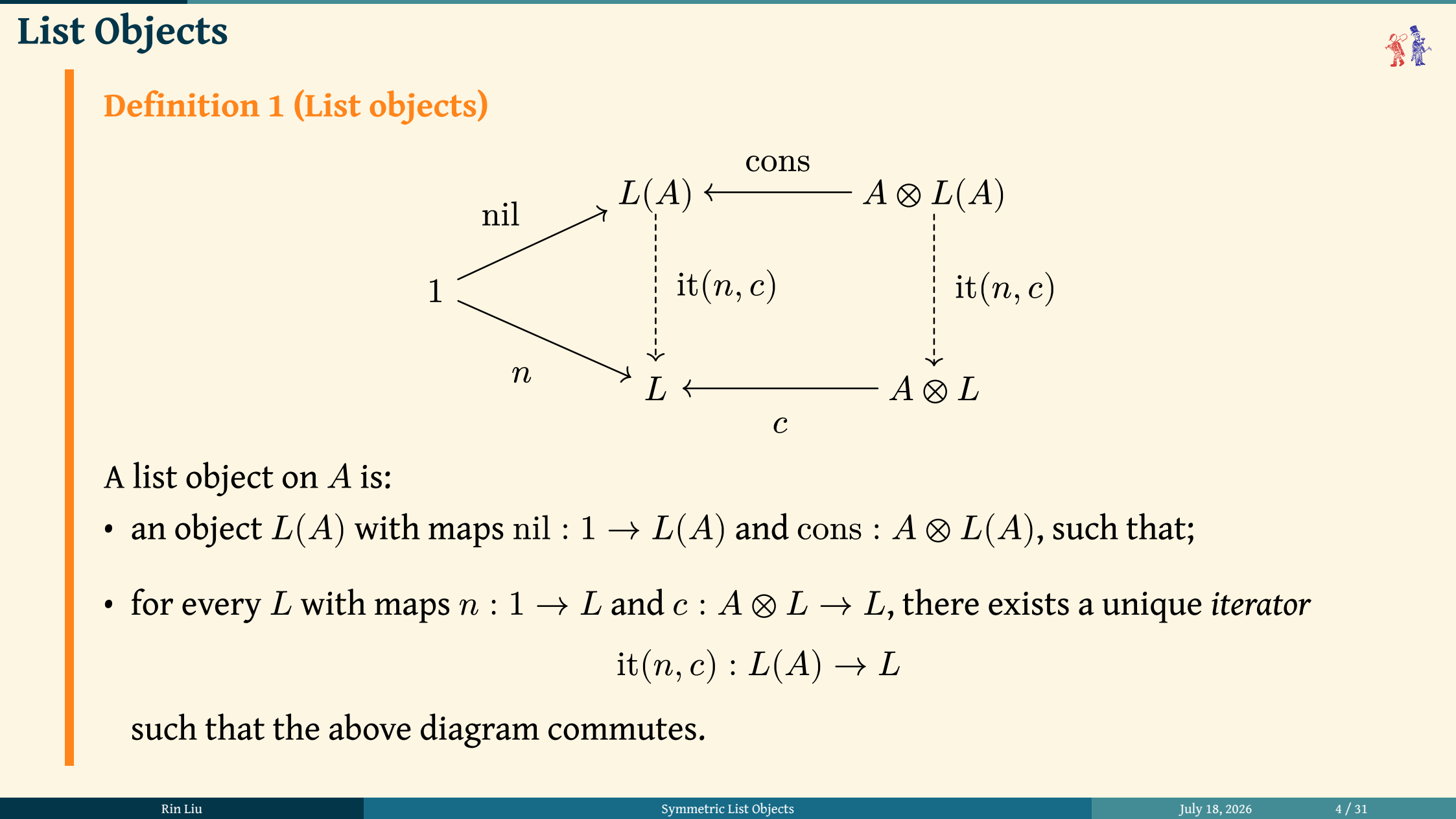

Definition 1 (List objects)

A list object on A is:

- an object $L(A)$ with maps $\mathrm{nil} : 1 \to L(A)$ and $\mathrm{cons} : A \otimes L(A)$, such that;
- for every L with maps $n : 1 \to L$ and $c : A \otimes L \to L$, there exists a unique *iterator*

$$\mathrm{it}(n, c) : L(A) \to L$$

such that the above diagram commutes.

Previous work

- Joyal introduced arithmetic universes with list objects
- Maietti[1] [2] studies them in type theory
- Paré and Roman[3] study monoidal categories with parametrized NNOs
- Cockett[4] and Jay[5] study pretoposes with list objects, called locoi
- Fiore and Saville[6] study parametrised list objects

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[1] M. E. Maietti, "Joyal's Arithmetic Universes via Type Theory," *Electronic Notes in Theoretical Computer Science*, vol. 69, pp. 272–286, Feb. 2003, doi: 10.1016/S1571-0661(04)80569-3.

[2] M. E. Maietti, "Joyal's Arithmetic Universe as List-Arithmetic Pretopos," *Theory and Applications of Categories*, vol. 24, pp. 39–83, 2010, doi: 10.70930/tac/qee78enb.

[3] R. Paré and L. Román, "Monoidal Categories with Natural Numbers Object," *Studia Logica*, vol. 48, no. 3, pp. 361–376, Sep. 1989, doi: 10.1007/BF00370829.

[4] J. Cockett, "List-Arithmetic Distributive Categories: Locoi," *Journal of Pure and Applied Algebra*, vol. 66, no. 1, pp. 1–29, Oct. 1990, doi: 10.1016/0022-4049(90)90121-W.

[5] C. Jay, "Finite Objects in a Locos," *Journal of Pure and Applied Algebra*, vol. 116, no. 1–3, pp. 169–183, Mar. 1997, doi: 10.1016/S0022-4049(97)81630-1.

[6] M. Fiore and P. Saville, "List Objects with Algebraic Structure," *LIPIcs, Volume 84, FSCD 2017*, vol. 84, p. 16:1–16:18, 2017, doi: 10.4230/LIPICS.FSCD.2017.16.

Examples of list objects in nature

- In $\mathsf{Set}(1, \times)$ (or any Grothendieck topos over Set), list objects are cons-lists:

$$1 \xrightarrow{\text{nil}} \mathcal{L}_f(A) \xleftarrow{\text{cons}} A \times \mathcal{L}_f(A)$$

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$$\text{Id} \xrightarrow{\text{nil}} F^*(F) \xleftarrow{\text{cons}} F \circ F^*(F)$$

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- In $\mathsf{Top}_\bullet(S^0, \wedge)$ (pointed spaces under smash product), list objects on connected spaces correspond to the James construction:

$$1 \xrightarrow{\text{nil}} J(A) \xleftarrow{\text{cons}} A \wedge J(A)$$

Limitations

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   we would need to curry the second argument to isolate the $L(A)$ on the left:

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2. These list objects also do not necessarily coincide with free monoids.
   - For example, every object of $\mathrm{Set}(0, +)$ is the free monoid on itself, but list objects on inhabited sets are always infinite:

     $$\mu X.0 + (A + X) \quad \cong \quad \mu X.A + X$$

Parametrised List Objects

To solve the first problem, we strengthen the universal property as follows:

Definition 2 (Parametrised list objects)

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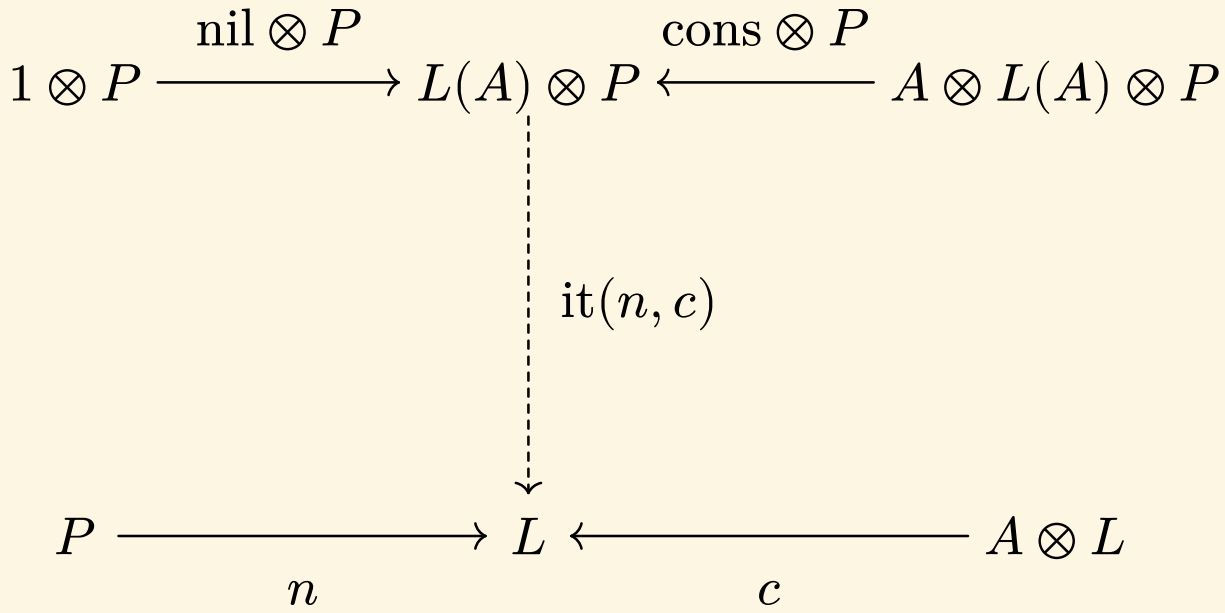

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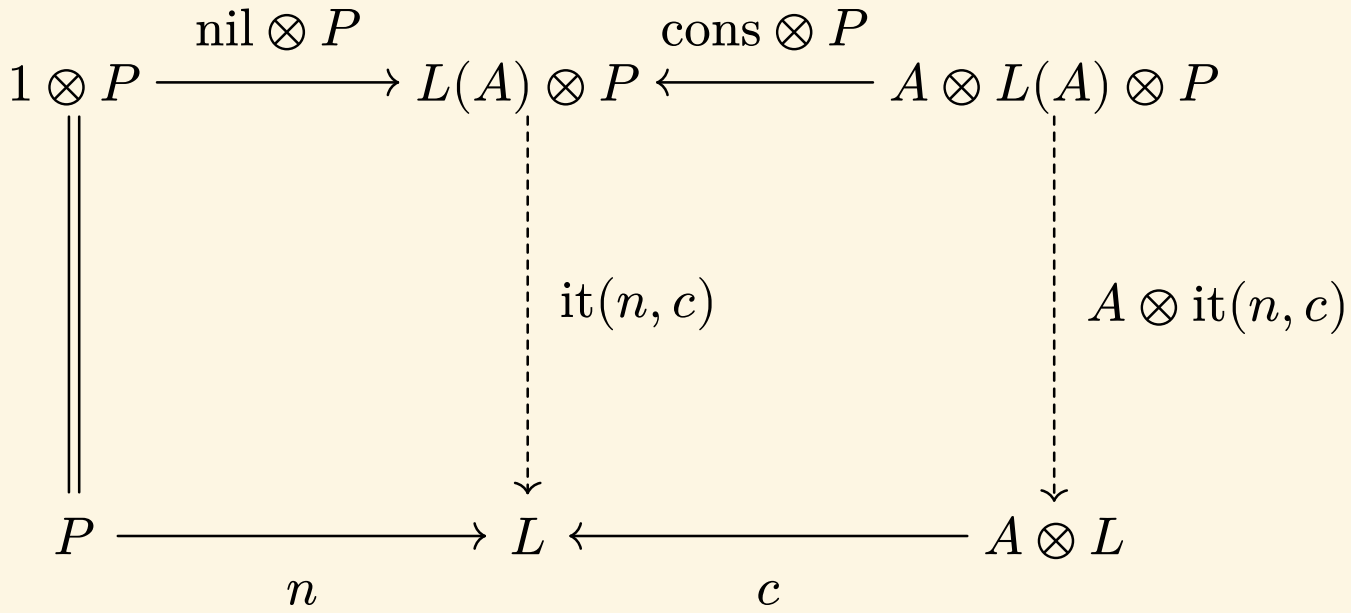
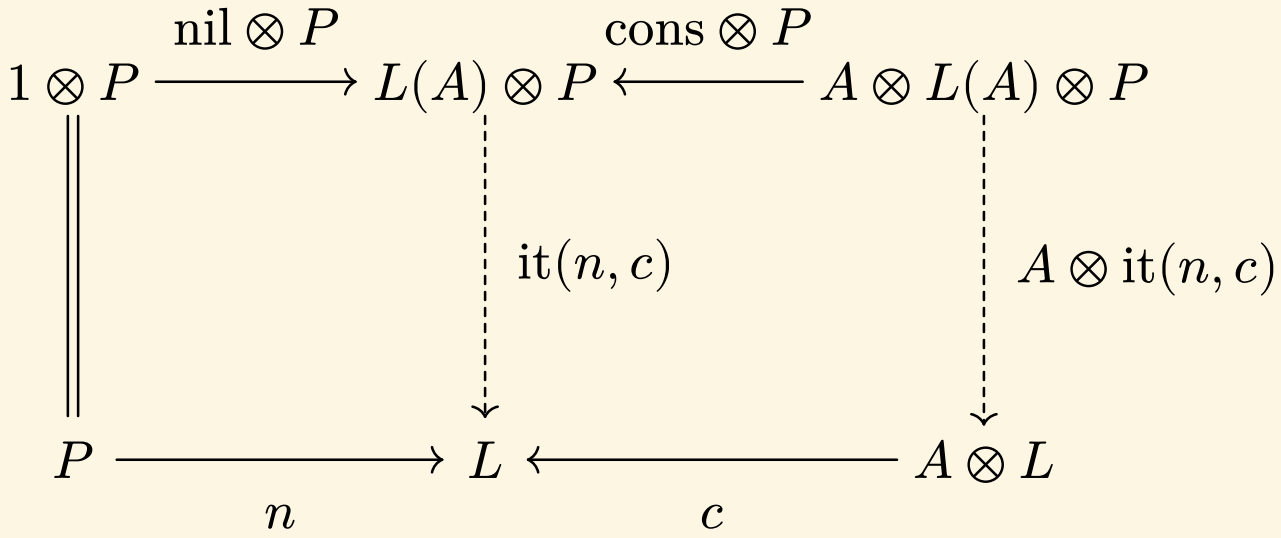
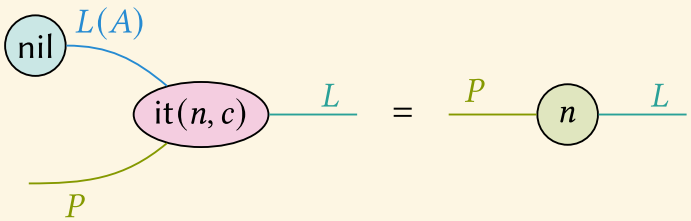
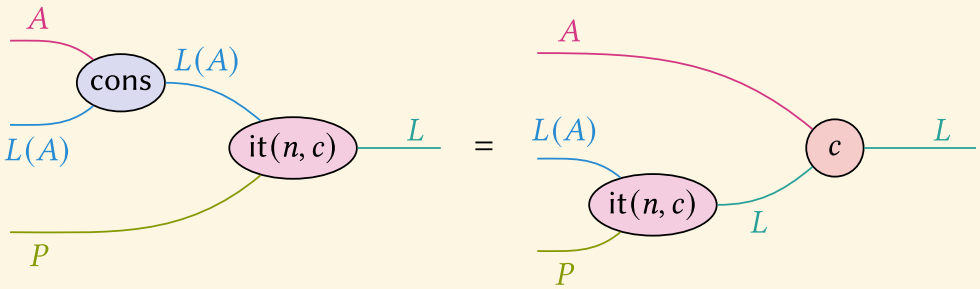


String Diagrams





(a) nil-β



(b) cons-β

Using the iterator

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$$\mathrm{it}(n, c) : L(A) \otimes P \to L$$

- Since $+\!+ : L(A) \otimes L(A) \to L(A)$, we have $L(A)$ as both the parameter P and the target L.
- We now only need maps $n : L(A) \to L(A)$ and $c : A \otimes L(A) \to L(A)$.

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```
_++_ : List A → List A → List A
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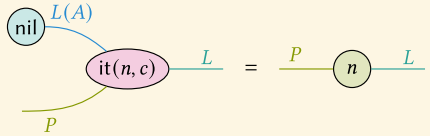

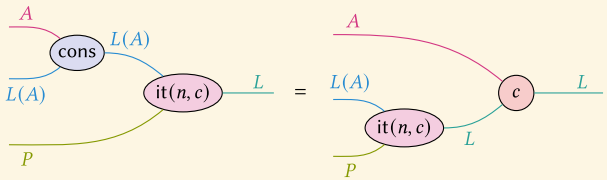

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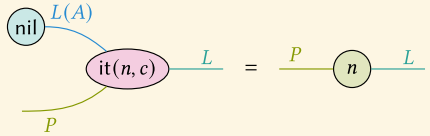

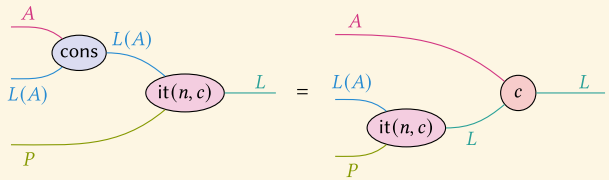

so we pick $n = \mathrm{id}$ and $c = \mathrm{cons}$:

$$+\!+ := \mathrm{it}(\mathrm{id}, \mathrm{cons}).$$

Monoids

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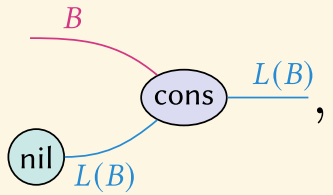
Lemma 2

If $\mathcal{C}$ admits parametrised list objects for every object, then $L(-)$ is a functor.

$$f : X \to Y \quad \mapsto \quad \left(X \xrightarrow{f} Y \xrightarrow{\eta_Y} L(Y)\right)^\sharp : L(X) \to L(Y)$$

$$\eta := \quad , \qquad \left(A \xrightarrow{g} (B, e, m)\right)^\sharp := \mathrm{it}(e, m \circ (g \otimes B))$$



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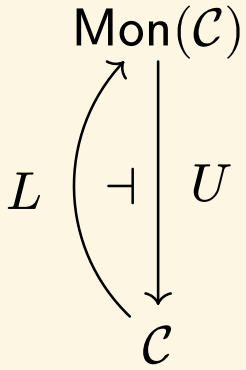

Theorem 3 (M. Fiore and P. Saville[1])

If $(\mathcal{C}, \times, 1)$ is cartesian monoidal, then the induced monad is strong.

[1] M. Fiore and P. Saville, "List Objects with Algebraic Structure," *LIPIcs, Volume 84, FSCD 2017*, vol. 84, p. 16:1–16:18, 2017, doi: 10.4230/LIPICS.FSCD.2017.16.

Symmetric Lists

Hereafter, the ambient category is symmetric strict monoidal.

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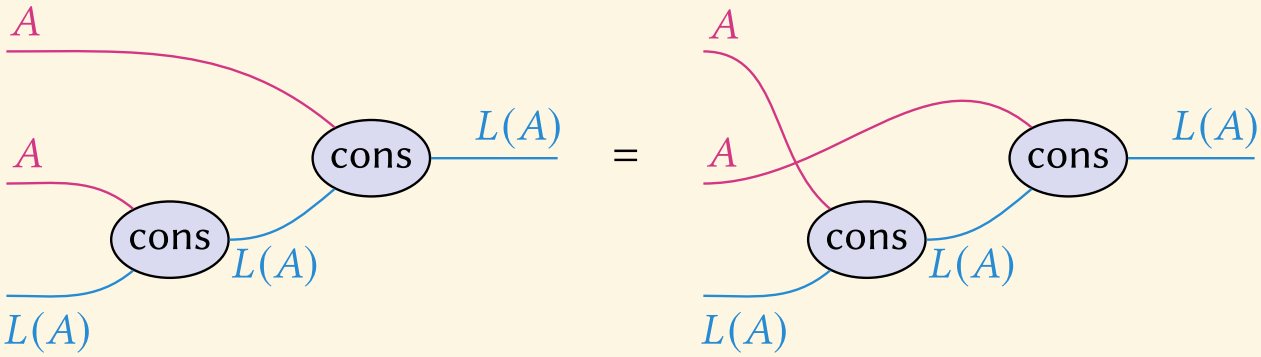

ilst axiom

A (*parametrised*) *symmetric list object* or *ilst* on A is a parametrised list object $L(A)$ satisfying the additional axiom above, and with the corresponding universal property.

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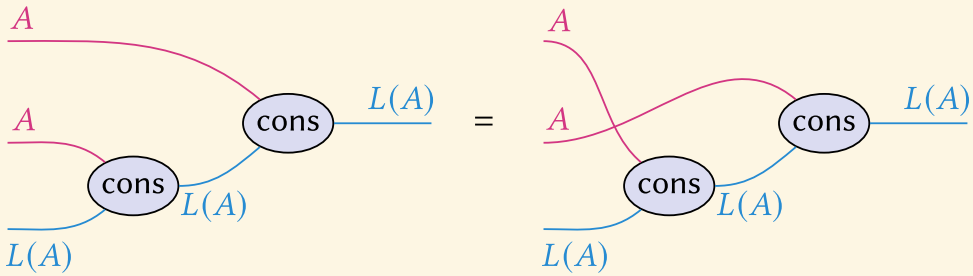


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  [] : ℐlst A
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     → a :: b :: xs ≡ b :: a :: xs
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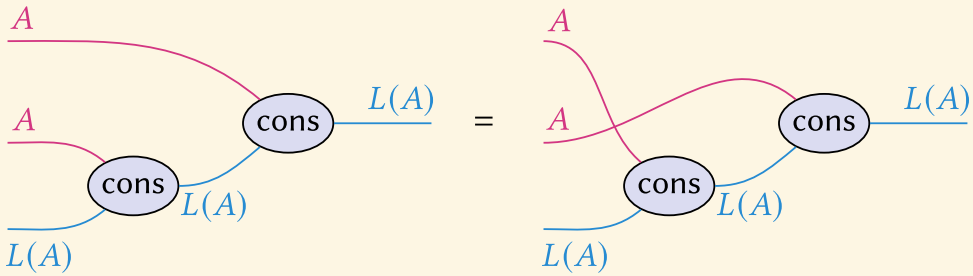


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$$\sum_{n \geq 0} A^n / S_n$$

Examples of symmetric list objects in nature

- In $\mathrm{Set}(1, \times)$ (or any Grothendieck topos over Set), symm. list objects are finite multisets:

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- In $\mathsf{Top}_\bullet(1, \times)$ (pointed topological spaces with cartesian product) symmetric list objects are the infinite symmetric product construction:

$$1 \xrightarrow{\text{nil}} \mathrm{SP}^\infty(X) \xleftarrow{\text{cons}} X \times \mathrm{SP}^\infty(X)$$

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- In $\mathsf{Rel}(1, \otimes)$ (binary relations with $\otimes$), symmetric list objects are finite multisets:

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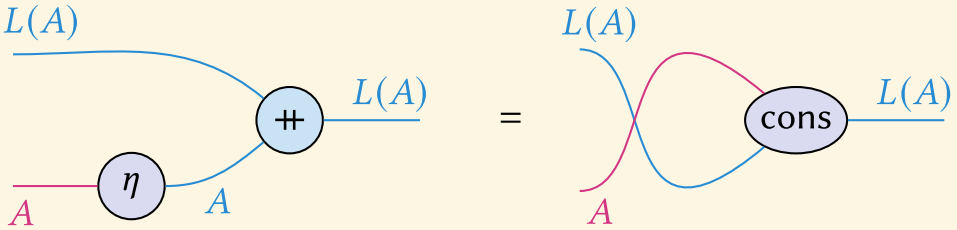
Symmetric Lists

Theorem 4

For any symmetric list $L(A)$:



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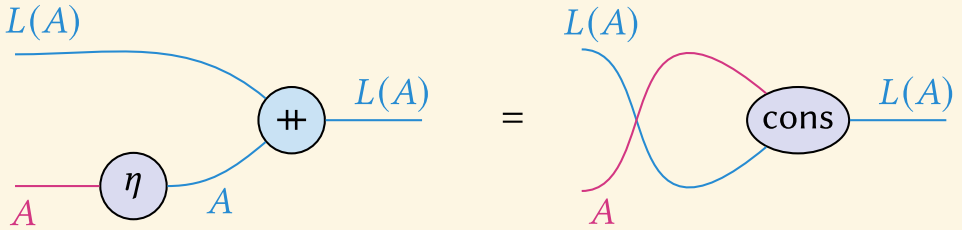

Proof. Let the two maps be f and g respectively.

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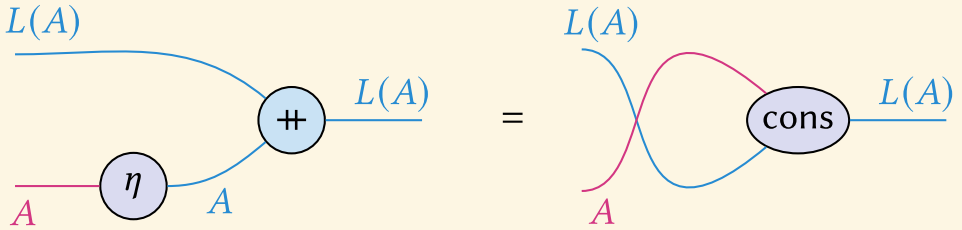

Proof. Let the two maps be f and g respectively.

We claim that $f = \mathrm{it}(\eta, \mathrm{cons}) = g$.

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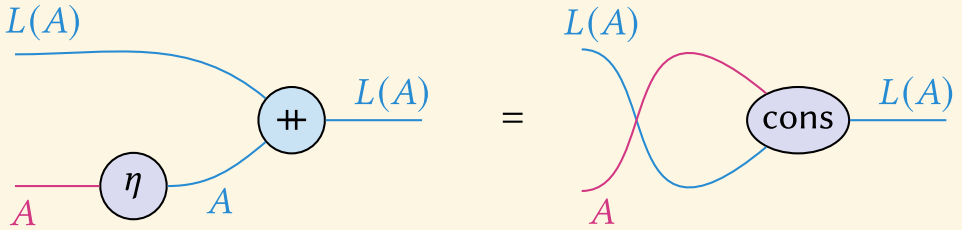
Theorem 4

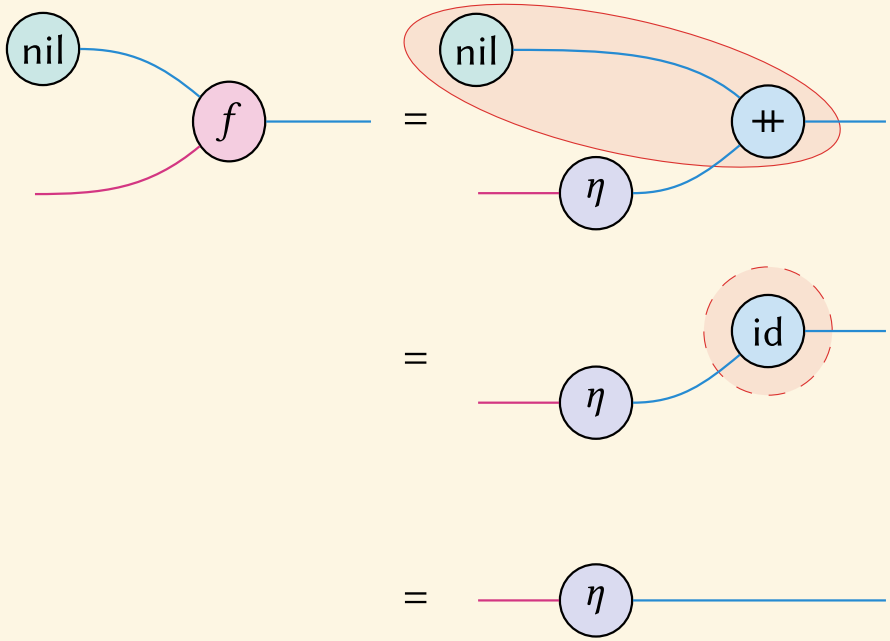
For any symmetric list $L(A)$:



Proof. Let the two maps be f and g respectively.

We claim that $f = \mathrm{it}(\eta, \mathrm{cons}) = g$.

Since the iterator is unique, it suffices to show that they both satisfy the two computation rules $\mathrm{nil}\text{-}\beta$ and $\mathrm{cons}\text{-}\beta$.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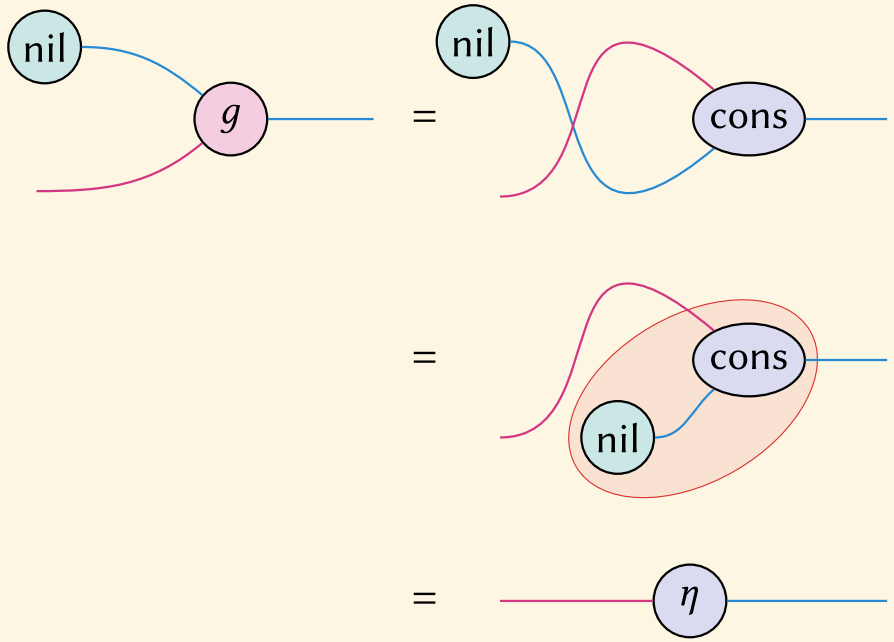
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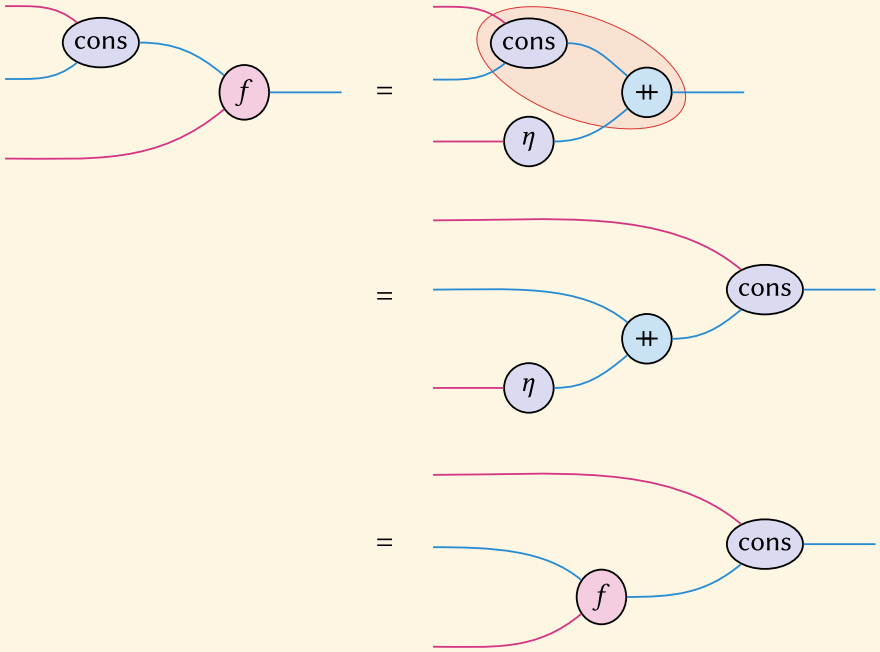
□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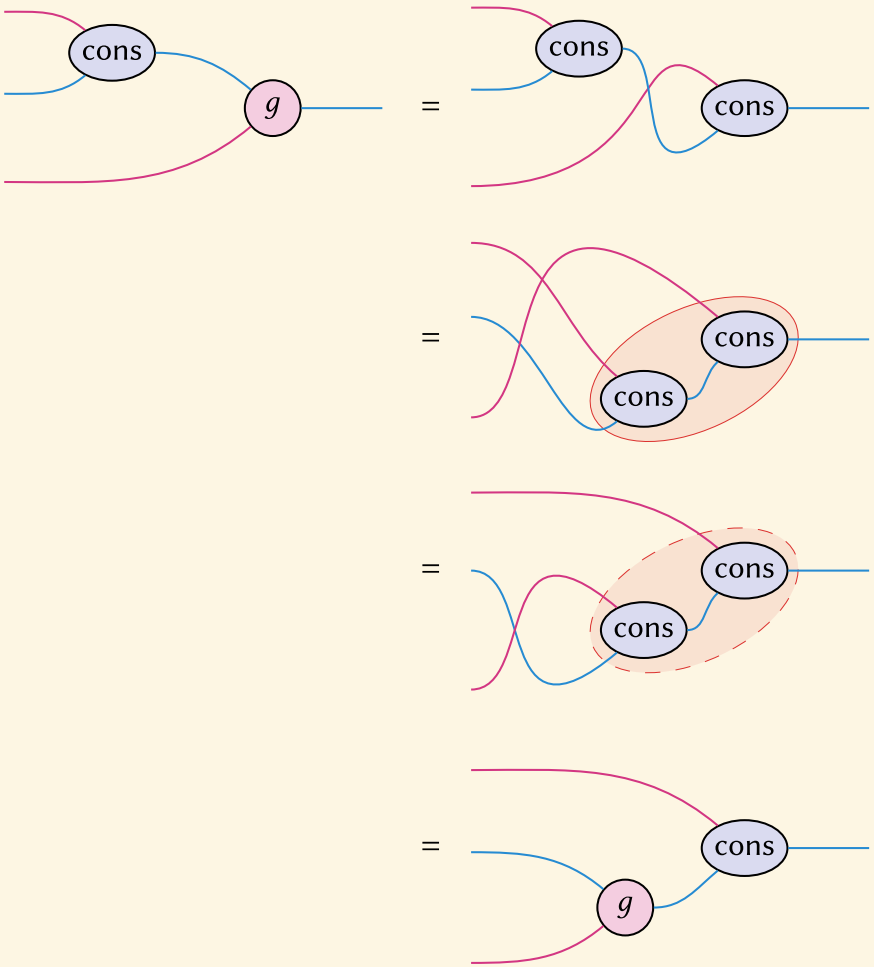
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□

Symmetric Lists

Theorem 5

Let $(L(A), \mathrm{nil}, \mathrm{cons})$ be a list object on A. Then, the following are equivalent:

1. $(L(A), \mathrm{nil}, \mathrm{cons})$ is a symmetric list object;
2. $(L(A), \mathrm{nil}, +\!\!+)$ is a commutative monoid object;
3. $(L(A), \mathrm{nil}, \mathrm{cons})$ is the free commutative monoid object on A.

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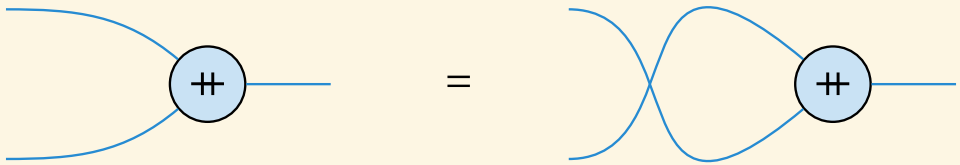
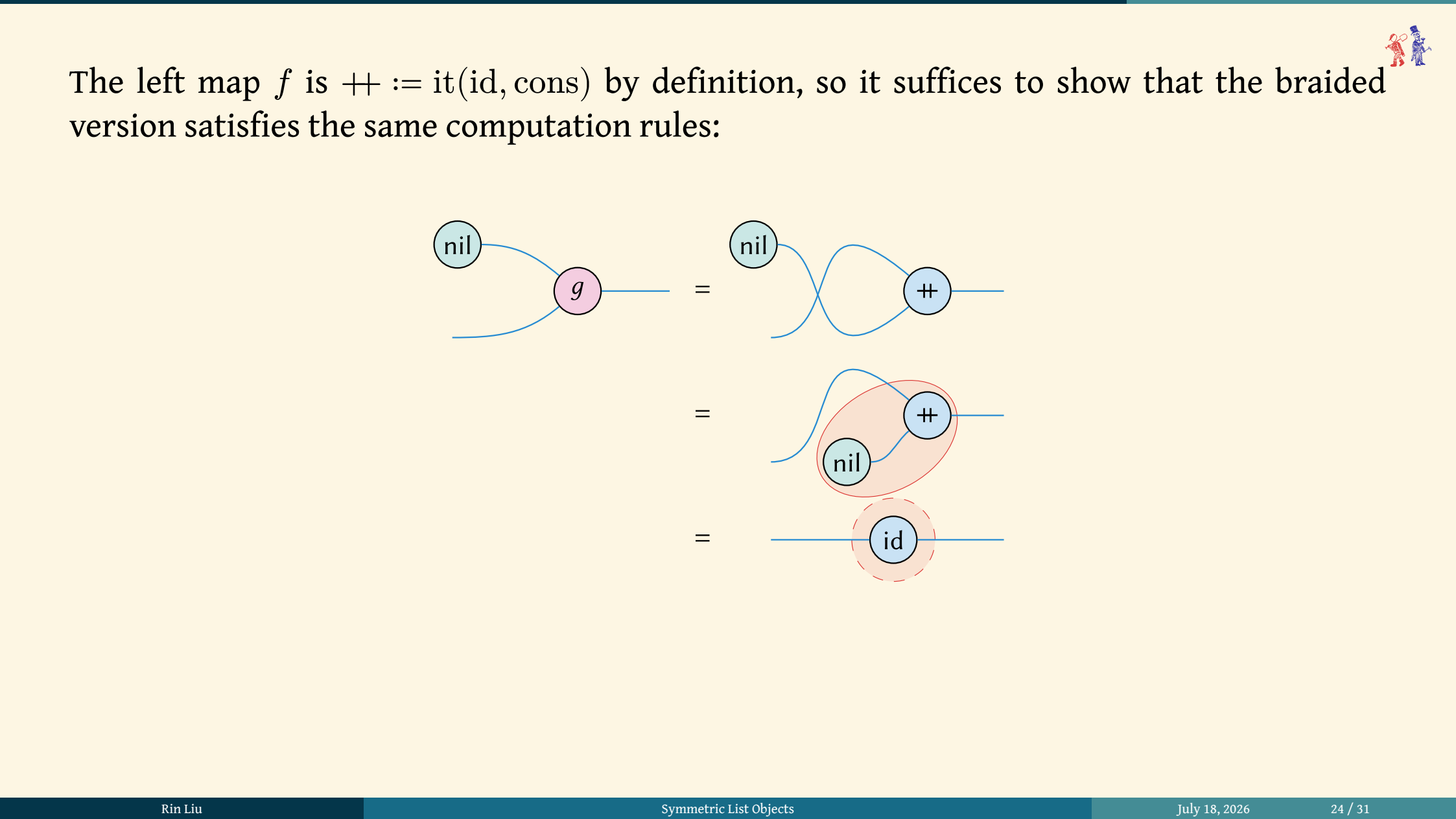
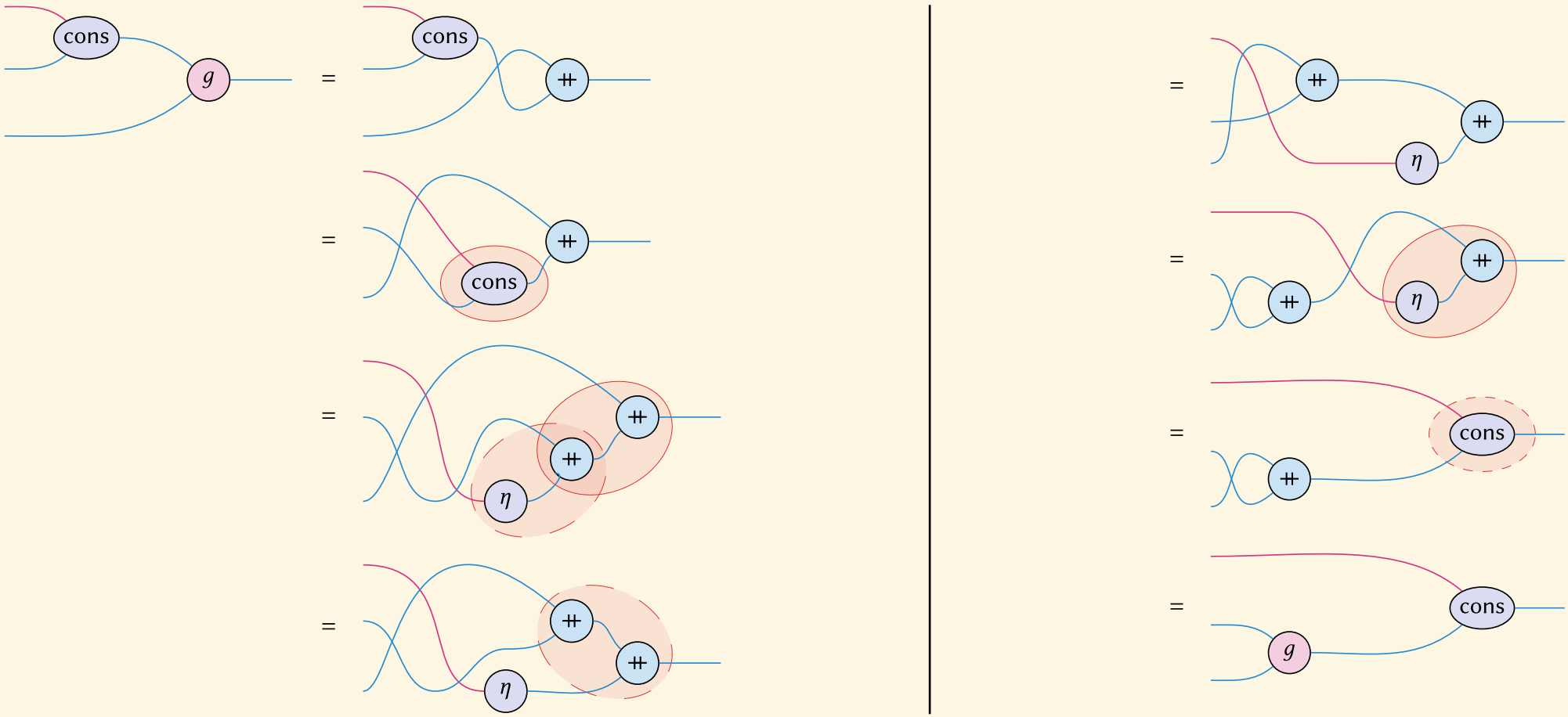
Proof. $(1 \Rightarrow 2)$ We already have that list objects are monoids, so we need only show that $+\!\!+$ is commutative:



Call the two maps f and g, respectively.



The left map f is $+\!\!+ := \mathrm{it}(\mathrm{id}, \mathrm{cons})$ by definition, so it suffices to show that the braided version satisfies the same computation rules: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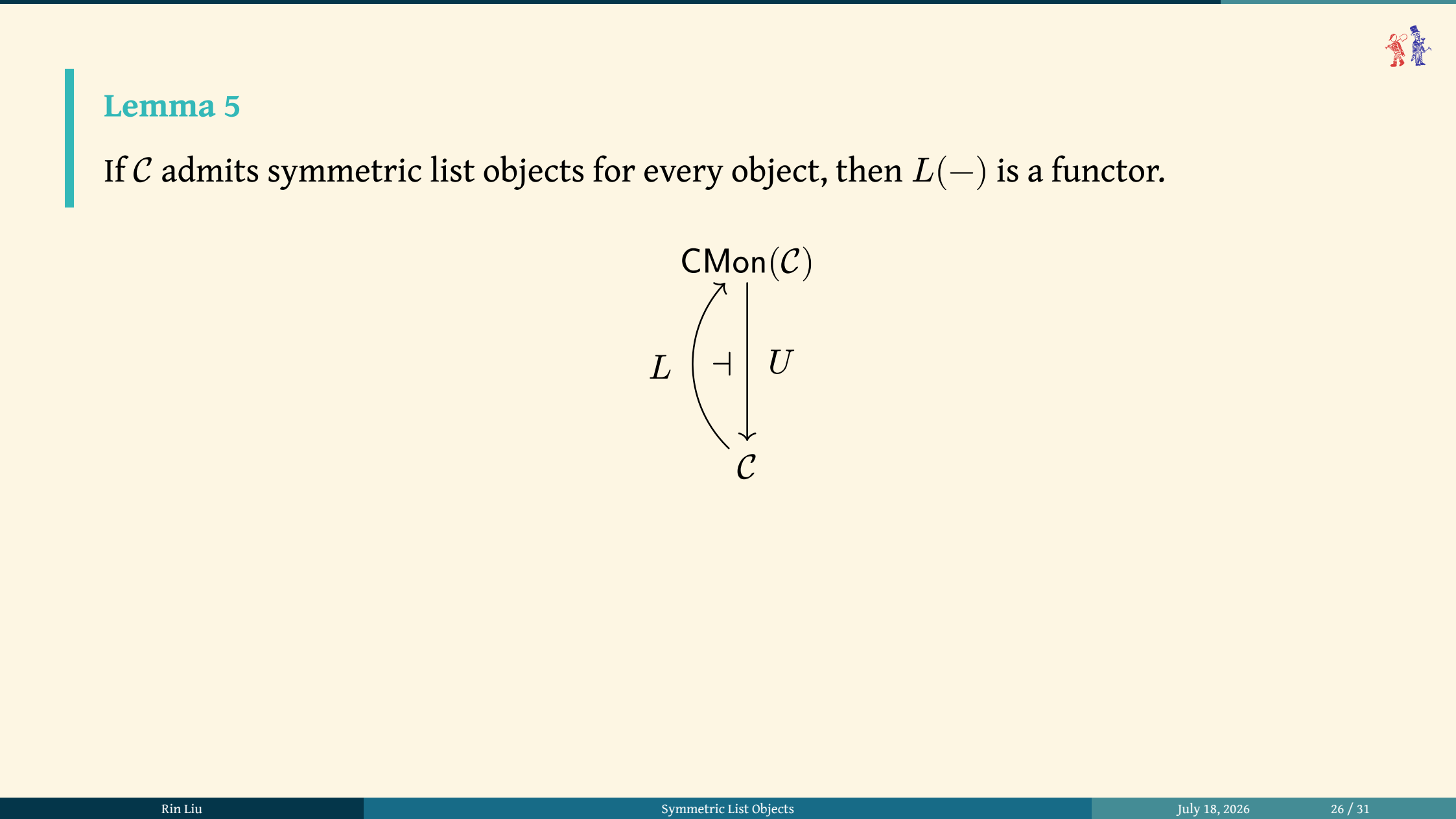
Lemma 5

If $\mathcal{C}$ admits symmetric list objects for every object, then $L(-)$ is a functor.



Lemma 5

If $\mathcal{C}$ admits symmetric list objects for every object, then $L(-)$ is a functor.

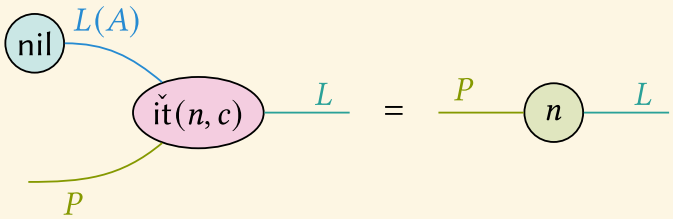
Theorem 6

If $(\mathcal{C}, \times, 1)$ is cartesian monoidal, and $L(A)$ is a list object, then $L(A)$ satisfies the following stronger universal property:

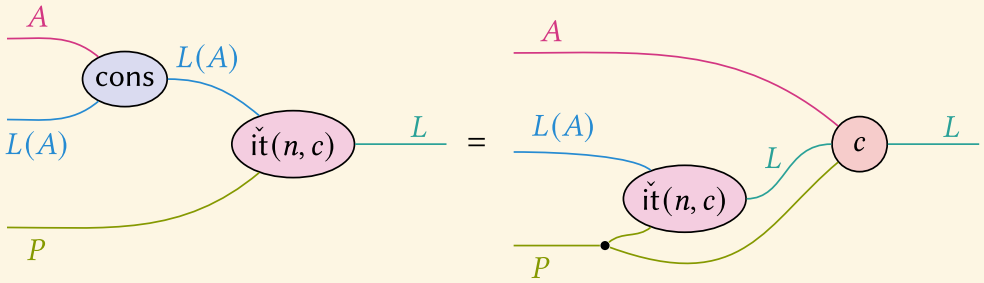
For every L with maps $P \xrightarrow{n} L \xleftarrow{c} A \times L \times P$, there exists a *doubly parametrised iterator*:

$$\check{\mathrm{it}}(n, c) : L(A) \times P \to L$$

such that the following computation rules hold:



(a) nil-β



(a) cons-β

Theorem 7

If $(\mathcal{C}, \times, 1)$ is cartesian monoidal, then the induced monad is strong, and further, commutative.

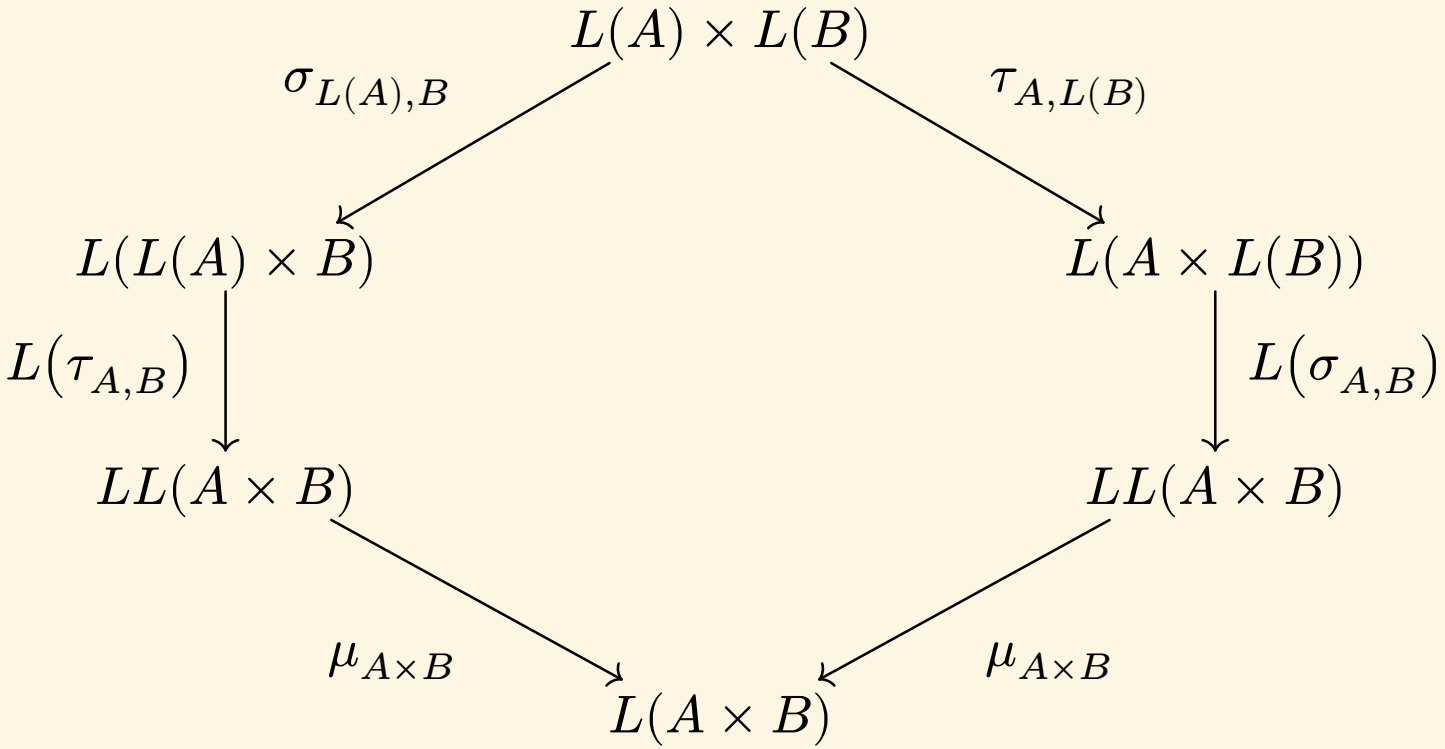
A construction of the right strength $\tau_{A,B} : L(A) \times B \to L(A \times B)$ is as follows.

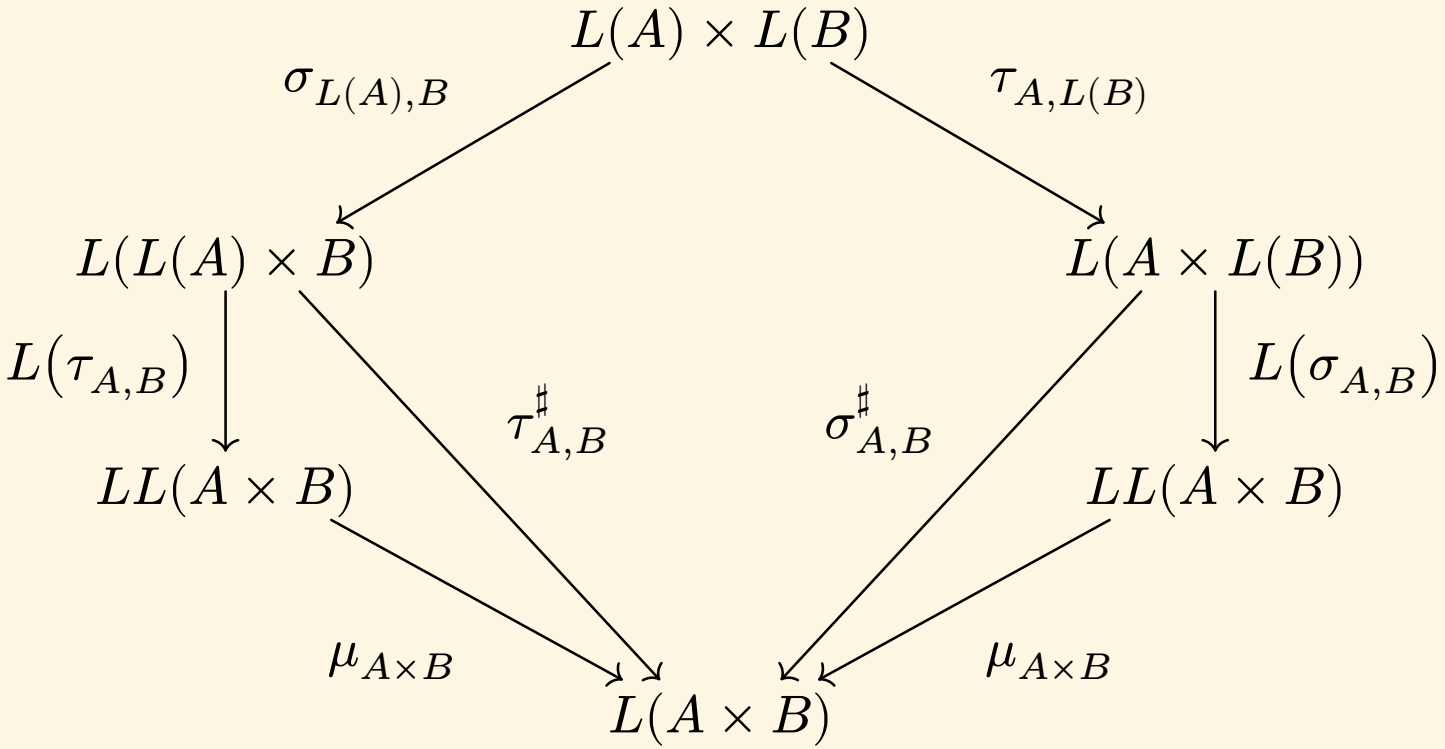
We can equip $L(A \times B)$ with a doubly-parametrised list algebra structure using:

- $n := B \xrightarrow{!} 1 \xrightarrow{\text{nil}} L(A \times B)$
- $c := A \times L(A \times B) \times B \xrightarrow{\cong} A \times B \times L(A \times B) \xrightarrow{\text{cons}} L(A \times B)$.

From the strong universal property, we obtain a map

$$\tau_{A,B} := \check{\text{it}}(n, c) : L(A) \times B \to L(A \times B)$$


$L(A) \times L(B)$
$\sigma_{L(A),B}$
$\tau_{A,L(B)}$
$L(L(A) \times B)$
$L(A \times L(B))$
$L(\tau_{A,B})$
$L(\sigma_{A,B})$
$LL(A \times B)$
$LL(A \times B)$
$\mu_{A\times B}$
$\mu_{A\times B}$
$L(A \times B)$


$L(A) \times L(B)$
$\sigma_{L(A),B}$
$\tau_{A,L(B)}$
$L(L(A) \times B)$
$L(A \times L(B))$
$L(\tau_{A,B})$
$L(\sigma_{A,B})$
$\tau^{\sharp}_{A,B}$
$\sigma^{\sharp}_{A,B}$
$LL(A \times B)$
$LL(A \times B)$
$\mu_{A\times B}$
$\mu_{A\times B}$
$L(A \times B)$

Conclusion

Summary

- Parametrised list objects give free monoids
- Symmetric list objects give free commutative monoids
- Iterator as recursion
- Doubly parametrised if Cartesian
- Strong monad if Cartesian
- Commutative monad if Symmetric

Future Work

- General theory of parametrised inital algebras
- Constructions of exponentials in LL
- Higher symmetric monoidal structures

Extended abstract with proofs