COMMUTATIVE CODENSITY MONADS AND PROBABILITY BIMEASURES

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1. Background

If $K\colon \mathcal{D} \to \mathcal{C}$ is any functor, then if the right Kan extension $\mathrm{Ran}_K K$ exists, it has a monad structure which is called the *codensity monad* of K. It was shown in [2] that several probability monads have presentations as pointwise codensity monads of functors from a small category of stochastic maps. More explicitly, we use the following four probability monads as examples:

(1) The Giry monad $\mathcal{G}$ on the category **Meas** of measurable spaces and measurable maps is the codensity monad of $K_{\mathcal{G}}\colon \mathbf{cStoch} \to \mathbf{Meas}$
(2) The countable expectation monad $\mathbb{E}$ on **Set** is the codensity monad of $K_{\mathbb{E}}\colon \mathbf{cStoch} \to \mathbf{Set}$
(3) The Radon monad $\mathcal{R}$ on the category **KHaus** of compact Haudorff spaces and continuous maps is the codensity monad of $K_{\mathcal{R}}\colon \mathbf{FinStoch} \to \mathbf{KHaus}$
(4) The Kantorovich monad $\mathcal{K}$ on the category **KMet** of compact metric spaces and 1-Lipschitz maps is the codensity monad of $K_{\mathcal{K}}\colon \mathbf{FinStoch} \to \mathbf{Meas}$

Where **cStoch** (resp. **FinStoch**) is the category of countable (resp. finite) sets and random functions between them. We give the interpretation that a functor $K_P\colon \mathbf{cStoch} \to \mathcal{C}$ defines a model of discrete probability in $\mathcal{C}$, and so its codensity monad P has a universal property as the largest extension of this model of discrete probability admissible in $\mathcal{C}$. We prove several properties of codensity monads, with the aim of exhibiting these properties in our examples. For example, if $\mathbb{T}$ is a monad on a category $\mathcal{C}$ with a terminal object 1 and coproducts, there is a monad structure on $T(1+-)$ which can be viewed as combining the computational effect modelled by T with possible errors. For a probability monad P, a Kleisli morphism of $X \to P(1+Y)$ is a stochastic map $X \nrightarrow Y$ with a probability of failure $p(\bot)$. If $\mathbb{T}$ is the pointwise codensity monad of K, then we are able to give conditions on K under which we also have a presentation $T(1+-) = \mathrm{Ran}_K K(1+-)$.

2. Commutativity of codensity monads

Motivated by the commutativity of probabilistic effects, we give a description of when codensity monads are commutative. It is a standard result, from [1], that commutative monads coincide with lax monoidal monads, and so we give general conditions for when codensity monads can be lifted to **MonCat**. We show that the condition $P(A\otimes B) = \lim_{(A\downarrow K_P)\times(B\downarrow K_P)} K_P(U_A \otimes U_B)$ is sufficient for our examples of probability monads P. Motivated by the theory of algebraic Kan extensions (see for example [4]), we call a codensity monad that satisfies the k-arity generalisation of this condition an *exactly pointwise monoidal* codensity monad. We characterise these as the monoidal codensity monads that factorise through $[\mathcal{D}, \mathbf{Set}]^{\mathrm{op}}$ in **MonCat** in a manner analogous to the factorisation of pointwise codensity monads in **Cat**. This also gives a description of the monoidal product in the Kleisli category of these monads in terms of Day convolution. For our examples, we show that $\lim_{\prod_{i=1}^k (A_i\downarrow K_P)} K_P(\bigotimes_{i=1}^k U_{A_i})$ is the space of *probability k-polymeasures*, and hence that $\mathcal{R}$, and the restriction of $\mathcal{G}$ to standard Borel spaces are exactly pointwise monoidal, but that our other examples are not.

We then introduce the notion of a $*$-*monad*, which behaves like a monad of several variables, and organises the categorical structure of the spaces of probability k-polymeasures. A $*$-monad is a structure that *lives in between* ordinary monads and commutative monads, and it acts on lists of objects and morphisms, i.e. there is an assignment $[a_1, .., a_k] \mapsto T[a_1, .., a_k]$. We provide a definition of a *monoidal op-multicategory*, and show that this is the structure of the Kleisli morphisms $Y \to T[X_1, .., X_k]$ of a $*$-monad T. Finally, we show that similarly to how any functor $K\colon \mathcal{D} \to \mathcal{C}$ from a small category $\mathcal{D}$ to a complete category $\mathcal{C}$ has a codensity monad $\mathbb{T}$, any such lax monoidal functor K has a codensity $*$-monad. For example the spaces of probability k-polymeasures form the codensity $*$-monad of $K_{\mathcal{G}}$, defining a monoidal op-multicategory of generalised stochastic maps.

3. Universal properties of probability monads

If $\mathbb{T}$ is a monad on $\mathcal{E}$ and $H\colon \mathcal{C} \to \mathcal{E}$, then if the right Kan lift $\mathrm{Rift}_H TH$ exists, it has a monad structure and additionally has a universal property as a lifting of $\mathbb{T}$ to $\mathcal{E}$. We show that our examples of probability monads have a related universal property as a lifting of $\mathcal{G}$, which generalises a result in [3], and we also show how it emerges from their codensity presentations. This gives a new connection between the measure theoretic definitions of these probability monads and their definitions as extensions of discrete probability. More explicitly, $\mathrm{Rift}_H TH$ is the terminal object of the category $(H_* \downarrow TH)$ whose objects (S, α) consist of an endofunctor S on $\mathcal{D}$ and a natural transformation $\alpha\colon HS \to TH$. Whenever $\mathbb{T}$ is a monad, $(H_* \downarrow TH)$ has a monoidal category structure such that monoids in $(H_* \downarrow TH)$ correspond to monads $\mathbb{S}$ on $\mathcal{C}$ admitting a *Kleisli law* to $\mathbb{T}$, i.e. a natural transformation $HS \to TH$ preserving the monad structure. Now if $\mathcal{C}$ is a category with a probability monad P then we might expect that there is a faithful functor $H\colon \mathcal{C} \to \mathbf{Meas}$ which assigns a measurable structure to objects of $\mathcal{C}$. Objects in $(H_* \downarrow \mathcal{G}H)$ can be interpreted as *endofunctors of probability measures on* $\mathcal{C}$ and the terminal object of a full monoidal subcategory of $(H_* \downarrow \mathcal{G}H)$ can be viewed as a *maximal monad of probability measures*. This idea was suggested by Van Breugel in [3], which introduced the monad $\mathcal{K}$ and showed it was the terminal object in a subcategory of $(H_* \downarrow \mathcal{G}H)$ for a suitable H. If $K_P\colon \mathbf{Stoch}_{\omega_1} \to \mathcal{C}$ additionally gives a model of discrete probability in $\mathcal{C}$ which is compatible with the measurable structure defined by H, then the codensity monad of K_P also has a universal property as *a maximal probability monad* in $\mathcal{C}$. We show that these two notions coincide when H and K_P satisfy certain compatibility conditions; a central condition being that $TH = \mathrm{Ran}_{K_P} K_T$. In particular we generalise Proosition 23 in [3] by showing that $\mathcal{K}$ is terminal in a larger monoidal subcategory of $(H_* \downarrow \mathcal{G}H)$, and we provide an analogous result for $\mathcal{R}$, giving a novel characterisation of this monad.

References

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