All your base categories are belong to us

A syntactic model of presheaves in type theory

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INRIA, Gallinette team

MSFP 2020

31st August 2020

It's Time to CIC Ass

CIC, the Calculus of Inductive Constructions.

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The Good Properties™

Consistency There is no proof of `False`.

Implementability Type-checking is decidable.

Canonicity Closed integers are indeed integers, i.e

$$\vdash M : \mathbb{N} \quad \text{implies} \quad M \equiv \mathtt{S} \ldots \mathtt{S}\,\mathtt{O}$$

Assuming we have a notion of reduction compatible with conversion:

Normalization Reduction is normalizing

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Some of these properties are interdependent

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Today we will focus on a specific family of models...

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PRESHEAVES!

- Proof-relevant Kripke semantics / Intuitionistic Forcing
- Bread and Butter of Model Construction
- They Are Everywhere: Cubical, Modal, Guarded, NbE, ...

I Herd U Liek Murphims

Definition

Let $\mathbb{P}$ be a category. A presheaf over $\mathbb{P}$ is just a functor $\mathbb{P}^{\mathsf{op}} \to \mathbf{Set}$.

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Bear with me, we will handwave through this in the next slides.

What is $\mathsf{Psh}(\mathbb{P})$?

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Objects: A presheaf $(\mathbf{A}, \theta_{\mathbf{A}})$ is given by

- A family of $\mathbb{P}$-indexed sets $\mathbf{A}_p : \mathbf{Set}$
- A family of "restriction morphisms" (a.k.a. monotonicity)

$$\theta_{\mathbf{A}} : \Pi\{p, q \in \mathbb{P}\}\,(\alpha \in \mathbb{P}(q, p)).\ \mathbf{A}_p \rightarrow \mathbf{A}_q$$

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s.t. given $x \in \mathbf{A}_p$, $\alpha \in \mathbb{P}(q, p)$ and $\beta \in \mathbb{P}(r, q)$:

$$\theta_{\mathbf{A}}\,\mathtt{id}_p\,x \equiv x \qquad\qquad \theta_{\mathbf{A}}\,(\beta \circ \alpha)\,x \equiv \theta_{\mathbf{A}}\,\beta\,(\theta_{\mathbf{A}}\,\alpha\,x)$$

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"Lowering is compatible with the structure of $\mathbb{P}$"

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- A family of $\mathbb{P}$-indexed functions $f_p : \mathbf{A}_p \to \mathbf{B}_p$
- which is natural, i.e. given $x \in \mathbf{A}_p$ and $\alpha \in \mathbb{P}(q, p)$

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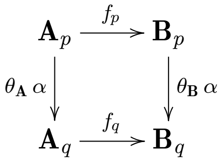
"f is compatible with restriction"



The Wise Speak Only of What They Know

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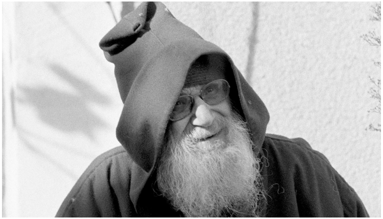

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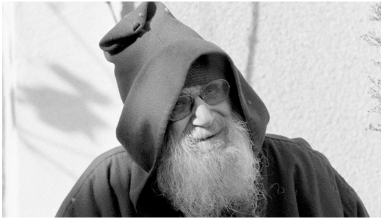

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Who cares?

Presheaves actually form a model of CIC.

$$\vdash A : \square \quad \rightsquigarrow \quad [\![A]\!] \in \mathsf{Psh}(\mathbb{P}) \qquad\qquad \vdash M : A \quad \rightsquigarrow \quad [M] \in \mathsf{Nat}(1, [\![A]\!])$$

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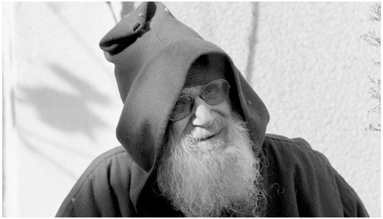

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Yet another *set-theoretical* model!

ZF Set Up Us The Bomb

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Phenomenological Law

Set-theoretical models suck.

Syntactic Models

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- Takes syntax as input.
- Interprets it into some low-level language.
- Must preserve the meaning of the source.
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"This is a *compiler*!"

On Curry-Howard Poetry

General models are more like **interpreters**.

No separation between target vs. host languages

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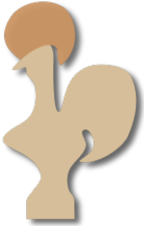
We will be interested in instances where $\mathcal{S}, \mathcal{T}$ are type theories.

Why Syntactic Models?

compilation →



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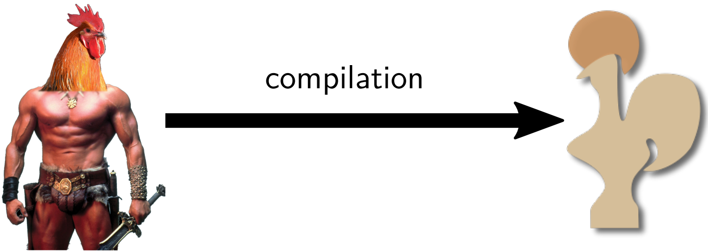

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Obviously, that's subtle.

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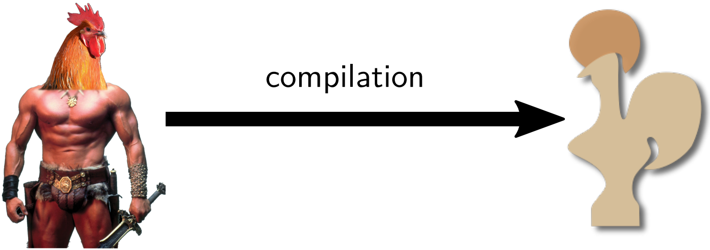

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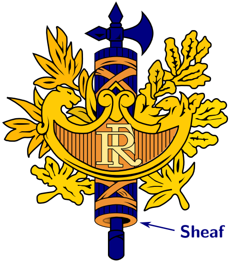
Yet, a lot of nice consequences.

- Does not require non-type-theoretical foundations (*monism*)
- If $\mathcal{T}$ is CIC, can be implemented in Coq (*software monism*)
- Inherit properties from CIC: computationality, decidability, **implementation**...

“Is it possible to see the presheaf construction as a syntactic model?”



French Coat of Arms

Persevere Diabolicum

Why the hell am I talking about syntactic presheaves today?

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2012	2016	2020
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FAIL	**FAIL**	**YAY?**

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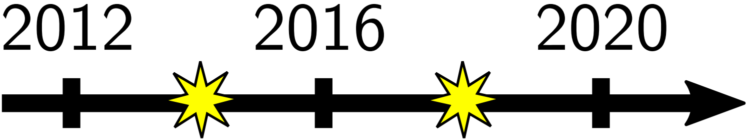

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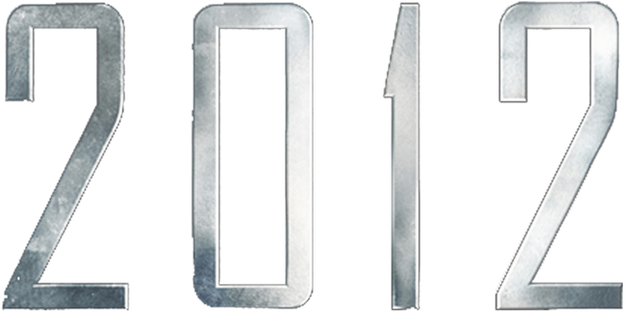
It is the journey, not the destination



(We were warned.)

Syntactic Presheaves, 2012 Edition

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What could possibly go wrong?

Close Encounters of the Third Type

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$$\texttt{Cat} : \square \quad := \quad \left\{ \begin{array}{l} \mathbb{P} : \square \\ \leq : \mathbb{P} \to \mathbb{P} \to \square \\ \texttt{id} : \Pi p.\, p \leq p \\ \circ : \Pi p\, q\, r.\, p \leq q \to q \leq r \to p \leq r \\ \texttt{eqn} : \ldots ; \end{array} \right\}$$

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And voilá, the Great Typification is an utter success!

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... except that equations are propositional !!!

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Thus the target theory must be EXTENSIONAL

That Was Not My Intension

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Syntactic models into ETT are not really syntactic models†.

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(†) To be more precise, I believe that ETT is not really a type theory.

2016

(Make conversion great again, and break everything else.)

Squaring the Circle

(Me to the authors of the 2012 paper, some time before defending PhD.)

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— What on earth does that even mean?

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Theorem

Kripke models factorize through CBPV.

$$\begin{array}{lcll} A & \text{value type} & \mapsto & [\![A]\!]^{\mathsf{v}} : \mathsf{Fun}(\mathbb{P}^{op}, \mathbf{Set}) \\ X & \text{computation type} & \mapsto & [\![X]\!]^{\mathsf{c}} : |\mathbb{P}| \to \mathbf{Set} \end{array}$$

This is the Left Adjoint, Right?

CBPV is a nice framework to study effects.

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$$[\![\mathcal{U}\ X]\!]^{\mathsf{v}}_p \quad := \quad \Pi(q : \mathbb{P})(\alpha : q \leq p).\,[\![X]\!]^{\mathsf{c}}_q \qquad \text{(free functoriality)}$$

$$\theta_{[\![\mathcal{U}\ X]\!]^{\mathsf{v}}}\ (\alpha : q \leq p)(x : [\![\mathcal{U}\ X]\!]^{\mathsf{v}}_p) := \lambda(r : \mathbb{P})(\beta : r \leq q).\, x\ r\ (\alpha \circ \beta)$$

More Than One Way to Do It

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Kripke models factorize through CBPV*.*

Canonical embeddings of λ-calculus into CBPV:

CBN	$(\sigma \to \tau)^{\mathsf{N}}$	$:=$	$\mathcal{U}\,\sigma^{\mathsf{N}} \to \tau^{\mathsf{N}}$	(a computation type)
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This is the presheaf interpretation of arrows! (up to naturality)**

Le Clash

Presheaves are *call-by-value*!

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In particular, they only satisfy the CBV equational theory generated by

$$(\lambda x.\, t)\ V \equiv_{\beta v} t\{x := V\}$$

because

$$t \equiv_{\beta v} u \quad \longrightarrow \quad t^{\mathsf{V}} \equiv_{\mathsf{CBPV}} u^{\mathsf{V}} \quad \longrightarrow \quad [t^{\mathsf{V}}]_p \equiv_{\mathcal{T}} [u^{\mathsf{V}}]_p$$

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Folklore

Call-by-name is not call-by-value!

If There is No Solution, There is No Problem

Easy solution! Pick the CBN decomposition instead.

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There is a syntactic "presheaf model" of CC^ω *into* CIC.

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$$\begin{array}{rcl} \vdash_{\mathsf{CC}^\omega} A : \Box & \longrightarrow & p : \mathbb{P} \vdash_{\mathsf{CIC}} [A]_p : \Pi(q : \mathbb{P})(\alpha : q \leq p).\, \Box \\ \vdash_{\mathsf{CC}^\omega} M : A & \longrightarrow & p : \mathbb{P} \vdash_{\mathsf{CIC}} [M]_p : [A]_p\ p\ \mathtt{id}_p \\ \vdash_{\mathsf{CC}^\omega} M \equiv N & \longrightarrow & p : \mathbb{P} \vdash_{\mathsf{CIC}} [M]_p \equiv [N]_p \end{array}$$

Robbing Peter to Pay Paul

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Not A Suprise

The Kripke translation introduces an effect! (a monotonic reader)

The Proverbial Paul

CBPV Folklore

- In effectful CBV, functions are not functions. (no substitution)
- In effectful CBN, inductive types are not inductive types. (no dep. elim.)

Conclusion of the Episode II

Good News

This is one of the first reasonable examples of dependent effects.

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Bad News

We still don't have a syntactic presheaf model.

Interlude

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In the meantime we worked quite a bit on effectful type theories

- Weaning translation
- Baclofen Type Theory
- Exceptional Type Theory
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This helped us understand what we first missed!

Values Are Not What They Once Were

Categorical presheaves form a model of the whole λ-calculus.

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- We do not have an equivalent in our CBN interpretation
- Isn't this some ad-hoc trick?

Completely Unrelated Slide

Consider an effectful CBV λ-calculus.

Definition (Führmann '99)

A term $t : A$ is said to be **thunkable** if it satisfies the equation

$$\texttt{let}\ x := t\ \texttt{in}\ \lambda().\,x \quad \equiv \quad \lambda().\,t$$

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Theorem (Folklore Realizability)

The sublanguage of hereditarily thunkable terms satisfies full β-conversion.

$$f \Vdash A \to B \quad := \quad \forall u. \quad u \Vdash A \quad \longrightarrow \quad f\ u\ \mathsf{thk} \quad \wedge \quad f\ u \Vdash B$$

Presheaves Are (Pure) Call-By-Value!

Now the magic trick.

Theorem

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Psh($\mathbb{P}$) is the "pure" subcategory of an effectful CBV language!

- This is a systematic construction that isn't tied to Kripke semantics.
- Unfortunately it relies on extensionality.
- What about CBN?

Syntactic Models For Free

A CBN counterpart of thunkability is **parametricity**

Bernardy-Lasson '11

There is a well-known parametricity interpretation for type theory

$$\Gamma \vdash_{\mathsf{CIC}} M : A \quad \longrightarrow \quad [\![\Gamma]\!]_\varepsilon \vdash_{\mathsf{CIC}} [M]_\varepsilon : [\![A]\!]_\varepsilon \; M$$

where $\quad [\![\cdot]\!]_\varepsilon := \cdot \quad$ and $\quad [\![\Gamma, x : A]\!]_\varepsilon := [\![\Gamma]\!]_\varepsilon, x : A, x_\varepsilon : [\![A]\!]_\varepsilon \; x$

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Turns out it is a syntactic model, compatible with intensionality!

It is a special case of a more general **internal realizability** interpretation.

On Parametric Presheaves

What does parametricity look like on the CBN presheaf model?

$$x : \mathbb{B} \quad \longrightarrow \quad \left\{ \begin{array}{l} x : (\Pi(q : \mathbb{P})(\alpha : q \leq p).\, \mathbb{B}) \\ x_\varepsilon : \mathbb{B}_\varepsilon\ p\ x \end{array} \right.$$

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But we also critically need to be compatible with the presheaf structure!

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😱 Guess what? The CBV vs. CBN conundrum is back. 😱

Trouble All The Way Up

This is exactly the CBV vs. CBN conundrum one level higher

Either you pick $\mathbb{B}_\varepsilon\ p\ x := (x = \lambda q\,\alpha.\,\mathtt{tt}) + (x = \lambda q\,\alpha.\,\mathtt{ff})$

$\rightsquigarrow$ this satisfies unicity but breaks definitionality (i.e. CBV).

Or you freeify $\mathbb{B}_\varepsilon\ p\ x := \Pi q\,\alpha.(\alpha \cdot x = \lambda r\,\beta.\,\mathtt{tt}) + (\alpha \cdot x = \lambda r\,\beta.\,\mathtt{ff})$

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It is not possible to get both at the same time in CIC!

Playing Cubes

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That is, the n-level proof is guaranteed to be pure by then $(n+1)$-level one.

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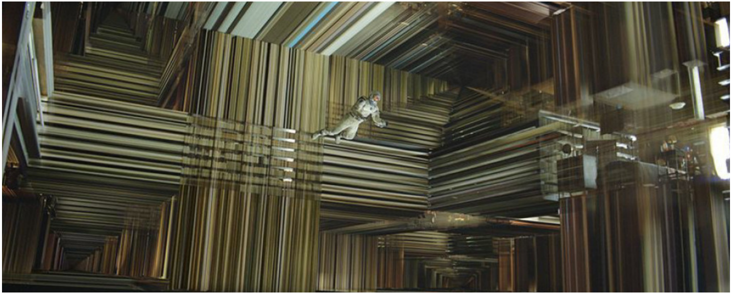

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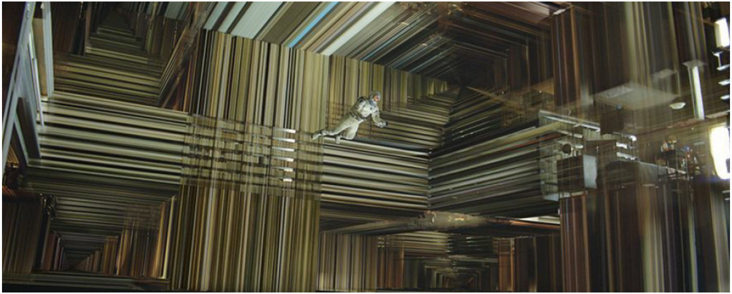

``Oh noes, not cubical type theory again!''

But CuTT itself is justified by presheaf models.

What would be the point to implement presheaves using presheaves?

2020

(On the virtues of Authoritarianism.)

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They introduce a new sort `SProp` of **strict propositions**.

$$M, N : A : \texttt{SProp} \quad \longrightarrow \quad \vdash M \equiv N$$

- A well-behaved subset of `Prop` compatible with HoTT
- It enjoys all good syntactic properties

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$\leadsto$ SProp is closed under products.

$$\vdash A : \square, \qquad x : A \vdash B : \texttt{SProp} \qquad \longrightarrow \qquad \vdash \Pi(x : A).\, B : \texttt{SProp}$$

$\leadsto$ Only False is eliminable from SProp into Type.

A Strict Doctrine

Possible Extension

$\mathfrak{s}$CIC additionally allows the elimination of `eq` from `SProp` to `Type`

This gives rise to a **strict equality**, i.e. $\mathfrak{s}$CIC has definitional UIP.

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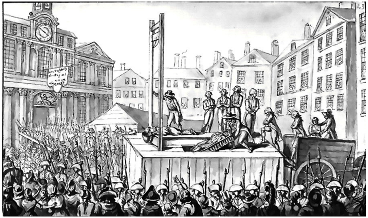
When the libertarian HoTT freely adds infinite towers of equalities...

... the authoritarian $\mathfrak{s}$CIC will instead **guillotine** all higher equalities.



Art. 1. *All humans are born uniquely equal in rights.*

Strict Parametricity

In the parametric presheaf translation

Strict equality is the authoritarian way to solve the coherence hell.

- make the parametricity predicate **free** $\leadsto$ **definitional functoriality**
- require it to be a **strict** proposition $\leadsto$ **proof uniqueness**

$$x : A \quad \longrightarrow \quad \begin{cases} x : \Pi(q \leq p).\, [\![A]\!]_q \\ x_\varepsilon : \Pi(q \leq p).\, [\![A]\!]_\varepsilon\ q\ (\alpha \cdot x) \end{cases}$$

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We call the result the **prefascist translation**. (lat. *fascis* : sheaf)

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We call the result the **prefascist translation**. (lat. *fascis* : sheaf)

Theorem

The prefascist translation is a syntactic model of CIC *into* $\mathfrak{s}$CIC.

- Full conversion, full dependent elimination.
- The actual construction is a tad involved, but boils down to the above.
- Unsurprinsingly, UIP is required to interpret universes (tricky!).

No Pain, No Gain

$\mathfrak{s}$CIC is way weaker than ETT

$\mathfrak{s}$CIC is **conjectured** to enjoy the usual good syntactic properties.

- Canonicity seems relatively easy to show
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- SN is problematic, e.g. $\mathfrak{s}$CIC + an impredicative universe is **not** SN
- Hoping that SN holds in the predicative case, decidability follows

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We don't rely on impredicativity in the prefascist model

We would inherit the purported good properties $\mathfrak{s}$CIC for free.

Back to Set

$\mathbf{Set}$ is a model of $\mathfrak{s}$CIC

Thus, the prefascist model can also be described set-theoretically.

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- a family of sets $\mathcal{A}_p$ for $p \in \mathbb{P}$.
- a family of predicates $(-) \Vdash_p \mathcal{A} \quad \subseteq \quad \mathsf{Cone}_p(\mathcal{A}) := \Pi(q : \mathbb{P})(\alpha : q \leq p).\, \mathcal{A}_q$

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A prefascist morphism f from $\mathcal{A}$ to $\mathcal{B}$ is

- a family of functions $f_p : \mathsf{El}_p\ \mathcal{A} \rightarrow \mathcal{B}_p$
- preserving predicates, i.e.

$$\forall x : \mathsf{El}_p\ \mathcal{A}. \quad \mathsf{app}_p(f, x) \Vdash_p \mathcal{B}$$

where

$$\begin{array}{lcl} \mathsf{El}_p\ \mathcal{A} & := & \{x : \mathsf{Cone}_p(\mathcal{A}) \mid \forall q\,(\alpha : q \leq p).\, (\alpha \cdot x) \Vdash_q \mathcal{A}\} \\ \mathsf{app}_p(f, x) & := & \lambda q\,(\alpha : q \leq p).\, f_q\ (\alpha \cdot x) \end{array}$$

Through The Looking Glass

Theorem

Prefascist sets over $\mathbb{P}$ form a category $\mathbf{Pfs}(\mathbb{P})$ with ***definitional*** *laws.*

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Takeaway: prefascist sets are a better presentation of presheaves

Application

Russian Constructivism

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Proofs are Kleene realizers

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Markov's Principle (MP)

$$\forall (f : \mathbb{N} \to \mathbb{B}).\, \neg\neg(\exists n : \mathbb{N}.\, f\, n = \mathtt{tt}) \to \exists n : \mathbb{N}.\, f\, n = \mathtt{tt}$$

- A lot of equivalent statements, e.g. a TM that doesn't loop terminates
- Semi-classical: $\mathbf{HA}^\omega \subsetneq \mathbf{HA}^\omega + \mathsf{MP} \subsetneq \mathbf{PA}^\omega$
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What if we tried to extend CIC with MP through a syntactic model?

Let's look at the realizer

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let mp f _ :=
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  while true do
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Proving `mp` $\Vdash$ MP needs MP in the meta-theory!

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We need something else...

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Not one, but at least **two** alternatives!

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CH's model is a mix of Kripke semantics and Friedman's A-translation

- Kripke semantics $\rightsquigarrow$ global cell $p : \mathbb{N} \rightarrow \mathbb{B}$ where

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The secret sauce is that the exception type depends on the current p

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Instead, we define the *Calculus of Constructions with Completeness Principles* as

$$\mathsf{CCCP} \quad (\supseteq \mathsf{CIC}) \quad \xrightarrow{\mathbf{Exn}} \quad \mathsf{CIC} + \mathcal{E} \quad \xrightarrow{\mathbf{Pfs}} \quad \mathfrak{s}\mathsf{CIC}$$

- $\mathbf{Pfs}$ is the prefascist model described before
- $\mathbf{Exn}$ is the exceptional model, a CIC-worthy A-translation

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Exn is a very simple syntactic model of CIC

Pick a fixed type $\mathcal{E}$ of **exceptions** in the target theory.

$$\vdash_{\mathcal{S}} A : \square \quad \longrightarrow \quad \vdash_{\mathcal{T}} [\![A]\!]_{\mathcal{E}} : \square \quad + \quad \vdash_{\mathcal{T}} [A]_{\mathcal{E}}^{\varnothing} : \mathcal{E} \to [\![A]\!]_{\mathcal{E}}$$

In particular $\quad [\![\neg A]\!]_{\mathcal{E}} \quad \cong \quad [\![A]\!]_{\mathcal{E}} \to \mathcal{E}$

Monic Fail

We perform the exceptional translation over an **exotic** type of exceptions

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We also have a modality in $\text{CIC} + \mathcal{E}$

$$\begin{array}{lcl} \texttt{local} & : & (\mathbb{N} \to \mathbb{B}) \to \square \to \square \\ [\texttt{local}\ \varphi\ A]_p & :\stackrel{\sim}{=} & [A]_{p \wedge \varphi} \end{array}$$

- $\texttt{return} : A \to \texttt{local}\ \varphi\ A$
- `local` commutes to arrows and positive types
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CCCP *validates* MP.

Proof by symbol pushing in CIC + $\mathcal{E}$ by the above and $[\![\neg A]\!]_{\mathcal{E}} \cong [\![A]\!]_{\mathcal{E}} \to \mathcal{E}$.

A Computational Analysis of MP

Every time we go under `local` we get new exceptions!

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The structure of the realizer thus follows closely Herbelin's proof.

$$\begin{array}{l} \texttt{mp}\ (p : \neg\neg(\exists n.\, f\ n = \texttt{tt})) := \\ \quad \texttt{try}_\alpha\ \bot_e\ (p\ (\lambda k.\, k\ (\lambda n.\, \texttt{raise}_\alpha\ n)))\ \texttt{with}\ \alpha\ n \mapsto n \end{array}$$

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Thus, Herbelin's proof is the direct style variant of Coquand-Hofmann

Final Digression

This is also highly reminiscent of NbE models

Two canonical ways to extend Kripke completeness to positive types:

- Add neutral terms to the semantic of positive types
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Two canonical ways to extend Kripke completeness to positive types:

- Add neutral terms to the semantic of positive types
- Add MP in the meta

Neutral terms behave as statically bound exceptions

As our model shows, this two techniques are morally equivalent.

This also highlights suspicious ties between delimited continuations and presheaves.

Conclusion

On presheaves:

- Presheaves are the pure fragment of an effectful CBV language
- We gave a computationally better-behaved presentation of presheaves
- It is a syntactic model that relies on strict equality in the target
- Provides for free extensions of CIC with SN, canonicity and the like
- ... assuming $\mathfrak{s}$CIC enjoys this $(\dagger)$

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TODO:

- Implement cubical type theory in this model

Thanks for your attention.